Chapter 8
Introduction to Hypothesis Testing

Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving a single population mean or proportion
- Formulate a decision rule for testing a hypothesis
- Know how to use the test statistic, critical value, and p-value approaches to test the null hypothesis
- Know what Type I and Type II errors are
- Compute the probability of a Type II error

What is a Hypothesis?

A hypothesis is a claim (assumption) about a population parameter:

- population mean
  
  Example: The mean monthly cell phone bill of this city is $µ = 42$

- population proportion

  Example: The proportion of adults in this city with cell phones is $p = .68$
The Null Hypothesis, $H_0$

- States the assumption (numerical) to be tested
  - Example: The average number of TV sets in U.S. Homes is at least three ($H_0: \mu \geq 3$)
- Is always about a population parameter, not about a sample statistic

The Null Hypothesis, $H_0$ (continued)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
  - Refers to the status quo
  - Always contains “=”, “≤” or “≥” sign
  - May or may not be rejected

The Alternative Hypothesis, $H_A$

- Is the opposite of the null hypothesis
  - e.g.: The average number of TV sets in U.S. homes is less than 3 ($H_A: \mu < 3$)
  - Challenges the status quo
  - Never contains the “=”, “≤” or “≥” sign
  - May or may not be accepted
  - Is generally the hypothesis that is believed (or needs to be supported) by the researcher
Hypothesis Testing Process

Claim: the population mean age is 50.
(Null Hypothesis: \( H_0: \mu = 50 \))

Is \( \bar{x} = 20 \) likely if \( \mu = 50 \)?

If not likely, REJECT Null Hypothesis

Reason for Rejecting \( H_0 \)

Sampling Distribution of \( \bar{x} \)

If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean...

... then we reject the null hypothesis that \( \mu = 50 \).

Level of Significance, \( \alpha \)

- Defines unlikely values of sample statistic if null hypothesis is true
- Defines rejection region of the sampling distribution
- Is designated by \( \alpha \), (level of significance)
- Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test
Level of Significance and the Rejection Region

- Level of significance = $\alpha$
- $H_0: \mu \geq 3$
- $H_A: \mu < 3$
  - Lower tail test
- $H_0: \mu \leq 3$
- $H_A: \mu > 3$
  - Upper tail test
- $H_0: \mu = 3$
- $H_A: \mu \neq 3$
  - Two tailed test

Errors in Making Decisions

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error
  - The probability of Type I Error is $\alpha$
    - Called level of significance of the test
    - Set by researcher in advance

Errors in Making Decisions (continued)

- **Type II Error**
  - Fail to reject a false null hypothesis
  - The probability of Type II Error is $\beta$
### Outcomes and Probabilities

**Possible Hypothesis Test Outcomes**

<table>
<thead>
<tr>
<th>State of Nature</th>
<th>Decision</th>
<th>H₀ True</th>
<th>H₀ False</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do Not Reject H₀</td>
<td>No error</td>
<td>(1 - α)</td>
<td>Type II Error</td>
</tr>
<tr>
<td>Reject H₀</td>
<td>Type I Error</td>
<td>(α)</td>
<td>No Error</td>
</tr>
</tbody>
</table>

**Key:**
- Outcome (Probability)

### Type I & II Error Relationship

- Type I and Type II errors cannot happen at the same time
  - Type I error can only occur if H₀ is true
  - Type II error can only occur if H₀ is false

If Type I error probability (α) ↑, then
Type II error probability (β) ↓

### Factors Affecting Type II Error

- All else equal,
  - β ↑ when the difference between hypothesized parameter and its true value ↓
  - β ↑ when α ↓
  - β ↑ when σ ↑
  - β ↑ when n ↓
Critical Value Approach to Testing

- Convert sample statistic (e.g., $\bar{x}$) to test statistic ($Z$ or $t$ statistic)
- Determine the critical value(s) for a specified level of significance $\alpha$ from a table or computer
- If the test statistic falls in the rejection region, reject $H_0$; otherwise do not reject $H_0$

Lower Tail Tests

- The cutoff value, $-z_\alpha$ or $\bar{x}_\alpha$, is called a critical value

Upper Tail Tests

- The cutoff value, $z_\alpha$ or $\bar{x}_\alpha$, is called a critical value

Two Tailed Tests

- There are two cutoff values (critical values):
  \[ \pm \frac{z_{\alpha/2}}{\sqrt{n}} \]

- Critical Value Approach to Testing

  - Convert sample statistic (\( \bar{x} \)) to a test statistic (\( Z \) or \( t \) statistic)
  - Hypothesis Tests for \( \mu \)
    - \( \sigma \) Known
    - \( \sigma \) Unknown
      - Large Samples
      - Small Samples

Calculating the Test Statistic

- The test statistic is:
  \[ Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \]
Calculating the Test Statistic

- **Hypothesis Tests for μ**
  - **σ Known**
  - **σ Unknown**

  **Large Samples**
  - The test statistic is:
  \[ t_{n-1} = \frac{\bar{x} - \mu}{s / \sqrt{n}} \]
  - But is sometimes approximated using a z:
  \[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \]

  **Small Samples**
  - (The population must be approximately normal)

Review: Steps in Hypothesis Testing

1. Specify the population value of interest
2. Formulate the appropriate null and alternative hypotheses
3. Specify the desired level of significance
4. Determine the rejection region
5. Obtain sample evidence and compute the test statistic
6. Reach a decision and interpret the result
Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is at least 3.
(Assume \( \sigma = 0.8 \))

1. Specify the population value of interest
   - The mean number of TVs in US homes
2. Formulate the appropriate null and alternative hypotheses
   - \( H_0: \mu \geq 3 \)    \( H_A: \mu < 3 \) (This is a lower tail test)
3. Specify the desired level of significance
   - Suppose that \( \alpha = .05 \) is chosen for this test

4. Determine the rejection region
   - \( \alpha = .05 \)
   - \( z_{\alpha} = -1.645 \)
   - This is a one-tailed test with \( \alpha = .05 \).
   - Since \( \sigma \) is known, the cutoff value is a z value:
     - Reject \( H_0 \) if \( z < z_{\alpha} = -1.645 \); otherwise do not reject \( H_0 \)

5. Obtain sample evidence and compute the test statistic
   - Suppose a sample is taken with the following results: \( n = 100, \overline{x} = 2.84 \) (\( \sigma = 0.8 \) is assumed known)
   - Then the test statistic is:
     \[
     z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}} = \frac{2.84 - 3}{0.8 / \sqrt{100}} = -1.16
     \]
Hypothesis Testing Example

6. Reach a decision and interpret the result

\[ z = -2.0 < -1.645 \]

Since \[ z = -2.0 < -1.645 \], we reject the null hypothesis that the mean number of TVs in US homes is at least 3.

An alternate way of constructing rejection region:

Now expressed in \( \bar{x} \), not \( z \) units

\[ \bar{x} = 2.84 < 2.8684 \]

Since \( \bar{x} = 2.84 < 2.8684 \), we reject the null hypothesis.

p-Value Approach to Testing

- Convert Sample Statistic (e.g. \( \bar{x} \)) to Test Statistic (\( Z \) or \( t \) statistic)
- Obtain the p-value from a table or computer
- Compare the p-value with \( \alpha \)
  - If p-value < \( \alpha \), reject \( H_0 \)
  - If p-value ≥ \( \alpha \), do not reject \( H_0 \)
p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme (≤ or ≥) than the observed sample value given $H_0$ is true
- Also called observed level of significance
- Smallest value of $\alpha$ for which $H_0$ can be rejected

Example: How likely is it to see a sample mean of 2.84 (or something further below the mean) if the true mean is $\mu = 3.0$?

$$P(\bar{x} < 2.84 \mid \mu = 3.0) = P\left(Z < \frac{2.84 - 3.0}{0.8} \right) = P(Z < -0.250) = 0.4013$$

Compare the p-value with $\alpha$
- If p-value < $\alpha$, reject $H_0$
- If p-value ≥ $\alpha$, do not reject $H_0$

Here: p-value = .028
$\alpha = .05$
Since .028 < .05, we reject the null hypothesis
Example: Upper Tail $z$ Test for Mean ($\sigma$ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over $52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

- $H_0: \mu \leq 52$ the average is not over $52 per month
- $H_a: \mu > 52$ the average is greater than $52 per month

(i.e., sufficient evidence exists to support the manager’s claim)

Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:

$\alpha = .10$

$z_{\alpha} = 1.28$

Reject $H_0$ if $z > 1.28$

Example: Find Rejection Region

Review: Finding Critical Value - One Tail

What is $z$ given $\alpha = 0.10$?

$\alpha = .10$

$z_{\alpha} = 1.28$

$\alpha = .10$

Standard Normal Distribution Table (Portion)

<table>
<thead>
<tr>
<th>Z</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>3790</td>
<td>3810</td>
<td>3830</td>
</tr>
<tr>
<td>1.2</td>
<td>3980</td>
<td>3997</td>
<td>4015</td>
</tr>
<tr>
<td>1.3</td>
<td>4147</td>
<td>4162</td>
<td>4177</td>
</tr>
</tbody>
</table>
Example: Test Statistic

Obtain sample evidence and compute the test statistic.

Suppose a sample is taken with the following results: \( n = 64, \ x = 53.1 \) (\( \sigma = 10 \) was assumed known).

Then the test statistic is:

\[
\frac{x - \mu}{\sigma / \sqrt{n}} = \frac{53.1 - 52}{10 / \sqrt{64}} = 0.88
\]

Example: Decision

Reach a decision and interpret the result:

\[
\alpha = .10
\]

\[ z = 0.88 \]

Do not reject \( H_0 \) since \( z = 0.88 \leq 1.28 \)

i.e.: there is not sufficient evidence that the mean bill is over $52

\[
p\text{-Value Solution}
\]

Calculate the p-value and compare to \( \alpha \)

\[
P(x \geq 53.1 | \mu = 52.0) = P \left( z \geq \frac{53.1 - 52}{10 / \sqrt{64}} \right)
= P(z \geq 0.88) = .1894
\]

Do not reject \( H_0 \) since \( p\text{-value} = .1894 > \alpha = .10 \)
Example: Two-Tail Test
(σ Unknown)

The average cost of a hotel room in New York is said to be $168 per night. A random sample of 25 hotels resulted in $\bar{x} = 172.50$ and $s = 15.40$. Test at the $\alpha = 0.05$ level.

$H_0: \mu = 168$
$H_A: \mu \neq 168$

(Assume the population distribution is normal)

Example Solution: Two-Tail Test

- $\alpha = 0.05$
- $n = 25$
- $\sigma$ is unknown, so use a $t$ statistic
- Critical Value:

$t_{24} \pm 2.0639$

Do not reject $H_0$: not sufficient evidence that true mean cost is different than $168$

Hypothesis Tests for Proportions

- Involves categorical values
- Two possible outcomes
  - “Success” (possesses a certain characteristic)
  - “Failure” (does not possess that characteristic)
- Fraction or proportion of population in the “success” category is denoted by $p$
Proportions

Sample proportion in the success category is denoted by \( \hat{p} \)

\[ \hat{p} = \frac{x}{n} \]

number of successes in sample

sample size

When both \( np \) and \( n(1-p) \) are at least 5, \( \hat{p} \) can be approximated by a normal distribution with mean and standard deviation

\[ \mu_{\hat{p}} = p \]

\[ \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} \]

(continued)

Hypothesis Tests for Proportions

The sampling distribution of \( \hat{p} \) is normal, so the test statistic is a \( z \) value:

\[ z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \]

\( \text{Hypothesis Tests for } p \)

\( np \geq 5 \) and \( n(1-p) \geq 5 \)

\( np < 5 \) or \( n(1-p) < 5 \)

Not discussed in this chapter

Example: \( z \) Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the \( \alpha = .05 \) significance level.

Check:

\( np = (500)(.08) = 40 \)

\( n(1-p) = (500)(.92) = 460 \)
Z Test for Proportion: Solution

\[ H_0: p = 0.08 \]
\[ H_A: p \neq 0.08 \]

\[ \alpha = 0.05 \]
\[ n = 500, \quad \hat{p} = 0.05 \]

Test Statistic:
\[ z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.05 - 0.08}{\sqrt{\frac{0.08(1-0.08)}{500}}} = -2.47 \]

Reject \( H_0 \) at \( \alpha = 0.05 \)

Conclusion:
There is sufficient evidence to reject the company's claim of 8% response rate.

\[ \text{p-value} = 0.0136 \]

\[ \text{p-value} < \alpha = 0.05 \]

Type II Error

\[ \text{Type II error is the probability of failing to reject a false } H_0 \]

Suppose we fail to reject \( H_0: \mu \geq 52 \) when in fact the true mean is \( \mu = 50 \)

Type II Error is the probability of failing to reject a false \( H_0 \)
Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu = 50$. This is the true distribution of $\bar{x}$ if $\mu = 50$. This is the range of $\bar{x}$ where $H_0$ is not rejected.

Type II Error

Here, $\beta = P(\bar{x} \geq \text{cutoff})$ if $\mu = 50$.

Calculating $\beta$

Suppose $n = 64$, $\sigma = 6$, and $\alpha = 0.05$.

$$\text{cutoff} = x_{\alpha} = \mu - z_{\alpha} \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = 50.766$$

So $\beta = P(\bar{x} \geq 50.766)$ if $\mu = 50$. 50.766
Calculating \( \beta \)

(continued)

Suppose \( n = 64, \sigma = 6, \) and \( \alpha = .05 \)

\[
P(x \geq 50.766 | \mu = 50) = P \left( z \geq \frac{50.766 - 50}{6/\sqrt{64}} \right) = P(z \geq 1.02) = .5 - .3461 = .1539
\]

Probability of type II error: \( \beta = .1539 \)

Using PHStat

Sample PHStat Output
Chapter Summary

- Addressed hypothesis testing methodology
- Performed $z$ Test for the mean ($\sigma$ known)
- Discussed $p$–value approach to hypothesis testing
- Performed one-tail and two-tail tests . . .

(continued)

Chapter Summary

- Performed $t$ test for the mean ($\sigma$ unknown)
- Performed $z$ test for the proportion
- Discussed type II error and computed its probability