Section 5.3 Permutations and Combinations

Permutations (order counts)

- A permutation of a set is an arrangement of the objects from a set.
- There are \( n! \) permutations of an \( n \)-element set, where \( 0! = 1 \) and \( n! = n \cdot (n - 1)! \).
- The number of permutations of length \( r \) chosen from an \( n \)-element set (where \( r \leq n \)) is
  \[
P(n, r) = \frac{n!}{(n-r)!}.
  \]

Example/Quiz. Let \( S = \{a, b, c\} \). Calculate each expression and list the corresponding permutations of \( S \): \( P(3, 3) \), \( P(3, 2) \), and \( P(3, 1) \).

Solution.

- \( P(3, 3) = 3!/0! = 6 \) with permutations \( abc, acb, bac, bca, cab, cba \).
- \( P(3, 2) = 3!/1! = 6 \) with permutations \( ab, ac, ba, bc, ca, cb \).
- \( P(3, 1) = 3!/2! = 3 \) with permutations \( a, b, \) and \( c \).

Quiz. Find the number of permutations of the letters in the word \( radon \).

Answer. \( 5! = 120 \).

Bag Permutations

The number of permutations of an \( n \)-element bag with \( k \) distinct elements, where the \( i \)th distinct element is repeated \( m_i \) times is

\[
\frac{n!}{m_1! \cdots m_k!}.
\]
The idea behind the formula
The idea is easy to see from an example. Suppose the bag is \([a, a, b, b, b]\). Then we can think of the letters as distinct elements of a set by placing subscripts on the repeated elements to obtain the set \(\{a_1, a_2, b_1, b_2, b_3\}\). There are 5! permutations of this set. But we don’t want to count permutations that are repeated if we drop the subscripts. For example, don’t want to count \(a_1 a_2 b_1 b_2 b_3\) and \(a_2 a_1 b_1 b_2 b_3\) as different. So we need to divide 5! by the number of permutations of each subscripted element: 2! for \(\{a_1, a_2\}\) and 3! for \(\{b_1, b_2, b_3\}\). This gives \(5!/(2!3!) = 10\) permutations of the bag \([a, a, b, b, b]\).

Example. Calculate the number of permutations of \([a, a, b, b]\) and list each permutation.
Answer. \(4!/(2!2!) = 6\) with permutations \(aabb, abab, abba, bbba, baba, baab\).

Quiz. Find the number of permutations of the letters in the word \(babbage\).
Answer. \(7!/(2!3!) = 420\).

Quiz (5 minutes). Find the smallest size \(n\) for strings of length \(n\) over \(\{a, b, c\}\) that can be used as distinct codes for 27 people, where \(a\) is repeated \(k\) times, \(b\) is repeated \(l\) times, \(c\) is repeated \(m\) times, and \(k + l + m = n\).
Answer. Use trial and error to solve the following inequality for the smallest \(n\) that satisfies the given conditions.
\[
\frac{n!}{k!l!m!} \geq 27
\]
The solution is \(n = 5\) with, for example, \(k = 2, l = 2,\) and \(m = 1\).
Combinations (order does not count)
The number of combinations of \( n \) things taken \( k \) at a time is the number of \( k \)-element subsets of an \( n \)-element set, and is given by

\[
C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

The idea: Take the number of permutations of \( n \) things taken \( k \) at a time. This count includes all the permutations of \( k \) elements. So divide by \( k! \) to get the count without regard to order.

Example/Quiz. Let \( S = \{a, b, c\} \). Calculate each expression and list the corresponding subsets of \( S \). \( C(3, 3) \), \( C(3, 2) \), \( C(3, 1) \), \( C(3, 0) \).

Solution. \( C(3, 3) = 1 \) with the subset \( \{a, b, c\} \). \( C(3, 2) = 3 \) with subsets \( \{a, b\} \), \( \{a, c\} \), \( \{b, c\} \). \( C(3, 1) = 3 \) with subsets \( \{a\} \), \( \{b\} \), \( \{c\} \). \( C(3, 0) = 1 \) with subset \( \emptyset \).

Quiz. Suppose there is a set of 5 cans of soda \( \{a, b, c, d, e\} \). Find the number of combinations of 5 cans of soda taken 3 at a time and list each combination.

Answer. \( C(5,3) = \binom{5}{3} = \frac{5!}{3!2!} = 10. \)

The 3-element subsets are \( \{a, b, c\} \), \( \{a, b, d\} \), \( \{a, b, e\} \), \( \{a, c, d\} \), \( \{a, c, e\} \), \( \{a, d, e\} \), \( \{b, c, d\} \), \( \{b, c, e\} \), \( \{b, d, e\} \), \( \{c, d, e\} \).

Binomial Theorem. \( (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \). Note: \( \binom{n}{k} \) is called a binomial coefficient.

Example.
\[
(x + y)^3 = \sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k = \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} xy^2 + \binom{3}{3} y^3 = x^3 + 3x^2 y + 3xy^2 + y^3.
\]
Bag Combinations
The number of \( k \)-element bags over an \( n \)-element set (with \( k \) and \( n \) positive) is given by

\[
{n + k - 1 \choose k}.
\]

The idea behind the formula
There is a bijection between the \( k \)-element bags over \( \{1, 2, \ldots, n\} \) and the \( k \)-element subsets of \( \{1, 2, \ldots, n, n + 1, \ldots, n + (k - 1)\} \). The bijection associates each \( k \)-element bag \([x_1, x_2, \ldots, x_k]\) where \( x_i \leq x_{i+1} \), with the \( k \)-element subset \([x_1, x_2+1, x_3 + 2, \ldots, x_k + (k - 1)]\), and the number of these \( k \)-element subsets is given by desired formula.

Example/Quiz. Let \( S = \{a, b, c\} \). Calculate the number of 3-element bags over \( S \) and list each bag.

Solution. \[ {3 + 3 - 1 \choose 3} = \binom{5}{3} = \frac{5!}{3!2!} = 10. \]

The 3-element bags are \([a, b, c], [a, b, b], [a, c, c], [a, a, b], [a, a, c], [a, a, a], [b, c, c], [b, b, c], [b, b, b], [c, c, c] \].

Quiz. Find the number of ways that 5 cans of soda can be chosen from a machine that dispenses 4 kinds of soda \( \{a, b, c, d\} \).

Solution. \[ {4 + 5 - 1 \choose 5} = \binom{8}{5} = \frac{8!}{3!5!} = 56. \] For example, \([a, a, b, b, d]\), and so on.