Preface

This manual is written to accompany *Discrete Structures, Logic, and Computability*, Third Edition, by James L. Hein. It contains the answers to all the exercises whose answers are not included in the book.

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Chapter 1

Section 1.1

2. An easy exercise.

3. b. The statement is false because \((3)(5) = 15\), which is odd. d. The statement is false. For example, 4 divides the product \((2)(6)\) but 4 does not divide 2 and 4 does not divide 6.

4. b. Let \(x\) be even and let \(y\) be odd. Then they can be written in the form \(x = 2m\) and \(y = 2n + 1\) for some integers \(m\) and \(n\). Therefore, the sum \(x + y\) can be written as \(x + y = 2m + 2n + 1 = 2(m + n) + 1\), which is an odd integer. d. We’ll prove the contrapositive statement: If \(n\) is odd then \(3n\) is odd. So if \(n\) is odd, then \(n = 2k + 1\) for some integer \(k\). It follows that \(3n = 3(2k + 1) = 6k + 3 = (6k + 2) + 1 = 2(3k + 1) + 1\), which is odd.

5. The converse is, “If \(x - y\) is even then \(x\) and \(y\) are odd.” This statement is false because, for example, \(4 - 2\) is even but neither 4 nor 2 is odd.

6. b. Let \(x = 5m + 6\), and let \(y = 5n + 6\) for some integers \(m\) and \(n\). Then \(xy = (5m + 6)(5n + 6) = 25mn + 30m + 30n + 36 = 5(5mn + 6m + 6n + 6) + 6\), which has the desired form.

7. b. Let \(d \mid (a + b)\) and \(d \mid a\). Then \(a + b = dk\) for some integer \(k\) and \(a = dn\) for some integer \(n\). Solving the first equation for \(b\) and substituting for \(a\) gives \(b = dk - a = dk - dn = d(k - n)\), which says that \(d\mid b\).

d. Let \(d \mid a\) and \(d \mid b\). Then there are integers \(k\) and \(j\) such that \(a = dk\) and \(b = dj\). So for any integers \(x\) and \(y\) we have \(ax + by = dkx + djy = d(kx + jy)\), which says that \(d \mid (ax + by)\).

8. b. First we’ll prove “If \(xy\) is odd then \(x\) is odd and \(y\) is odd,” by proving the contrapositive “if \(x\) is even or \(y\) is even then \(xy\) is even.” If \(x\) is even, then \(x = 2k\) for some integer \(k\). So \(xy = 2ky\), which is even. Next we’ll prove “If \(x\) is odd and \(y\) is odd then \(xy\) is odd.” If \(x\) and \(y\) are odd, then \(x = 2m + 1\) and \(y = 2n + 1\) for some integers \(m\) and \(n\). So \(xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1\), which is odd. So the iff statement is proven.

d. First we’ll prove “If \(m \mid n\) and \(n \mid m\) then \(n = m\) or \(n = -m\).” If \(mln\) and \(nlm\), then \(n = mk\) and \(m = nj\) for some integers \(k\) and \(j\). Substituting for \(m\) in the first equation gives \(n = mk = njk\). Cancellation of \(n\) gives \(1 = jk\), which implies that either \(j = k = 1\) or \(j = k = -1\). Therefore, either \(n = m\) or \(n = -m\). Next we’ll prove “If \(n = m\) or \(n = -m\) then \(m \mid n\) and \(n \mid m\).” If \(n = m\), then \(n = m(1)\) and \(m = n(1)\), so \(m \mid n\) and \(n \mid m\). If \(n = -m\), then \(n = m(-1)\) and \(m = n(-1)\), so \(m \mid n\) and \(n \mid m\). Thus the iff statement is proven.
10. Let $3 \mid x$ and $5 \mid x$. Then $x = 3m = 5n$ for some integers $m$ and $n$. We can write $3m = 5n = 3n + 2n$. Solve for $2n$ to obtain $2n = 3(m - n)$. This says that $3 \mid 2n$. Now proceed as in Example 1.2 to obtain $3 \mid n$. So $n = 3k$ for some $k$. Therefore, $x = 5n = 5(3k) = 15k$ so that $15 \mid x$.

Section 1.2

1. b. $\{5, 9, 13, 17\}$. d. $\{\text{January, February, May, July}\}$.
   f. $\{1, 2, 3, 4, 6, 12, 24\}$.

2. b. $\{x \mid x = 2k + 1 \text{ and } k \in \mathbb{N} \text{ and } 0 \leq k \leq 7\}$ or $\{2k + 1 \mid k \in \mathbb{N} \text{ and } 0 \leq k \leq 7\}$
   d. $\{x \mid x = 2k \text{ and } k \in \mathbb{Z}\}$ or $\{2k \mid k \in \mathbb{Z}\}$.


4. If the statement is false, then there must be some element in $\emptyset$ that is not in $A$. But this can’t happen since $\emptyset$ doesn’t have any elements. Therefore, the statement must be true.

6. b. $\{\emptyset, \{a\}, \{\{a, b\}\}, \{a, \{a, b\}\}\}$
   d. $\{\emptyset, \emptyset\}$.

7. b. $\{a, \emptyset\}$. d. $\{a, b, \{b\}\}$.

9. b. $x = 6$. d. $x = 7$.

13. b. $\{0, 1, 6, 7, 8, 9\}$. d. $\{6, 7, 8, 9\}$. f. $\{0, 1, 4, 5, 6, 7, 8, 9\}$.

14. b. $B \cup C - A \cap C$.

15. b. Yes, the equality holds.

   d. The equality follows by noticing two things. First, if $x$ is in the given union, then $x \in D_{2k}$ for some $k \geq 1$. So $x$ must be a nonzero natural number. Second, if $x$ is a nonzero natural number, then $x$ divides $2x$. Therefore, $x \in D_{2x}$ and it follows that $x$ is in the union. Therefore, $\bigcup_{k=1}^{\infty} D_{2k} = \mathbb{N} \setminus \{0\}$.

16. b. Yes, the equality holds.

   d. The equality follows by noticing two things. First, if $x$ is in the given union, then $x \in M_{2k+1}$ for some $k \geq 1$. So $x$ is either $0$, in which case it is not a power of $2$, or $x$ can be written as $x = n(2k + 1)$ for some nonzero natural number $n$, in which case it is divisible by an odd number and thus cannot be a power of $2$. Second, if $x$ is a natural number that is not a power of $2$ and $x \neq 0$, then it is divisible be an odd number of the form $2k + 1$ with $k \geq 1$. So we can write $x = n(2k + 1)$ for some nonzero natural number $n$, which tells us that $x$ is in the given union. Therefore, $\bigcup_{k=1}^{\infty} M_{2k+1} = \mathbb{N} \setminus \{2^n \mid n \in \mathbb{N}\}$.

18. 50.

19. b. 19. d. 16.

21. At least 50 percent of the population visited both.
23. The answer is zero. Let $A$, $B$, and $C$ be sets of people attending each session, with cardinalities 15, 18, and 12, respectively. Since the total number attending the conference is 25, it follows that $|A \cap B| \geq 8$. So we can consider the set $(A \cap B) \cup C$, which we can use to find the minimum value for $|A \cap B \cap C|$ as follows: $25 \geq |(A \cap B) \cup C| = |A \cap B| + |C| - |A \cap B \cap C| = 15 + 12 + |C| - |A \cap B \cap C|$. Solving for $|A \cap B \cap C|$ we obtain $|A \cap B \cap C| \geq 8 + 12 - |A \cap B \cap C|$. Since $|A \cap B \cap C|$ is nonnegative, it follows that $|A \cap B \cap C| \geq 0$. So it is possible that no one attended all three sessions. For an alternative solution use complements and De Morgan’s laws.

24. b. $[x, x, y, y], [x, y, x]$. d. $[1, 2, 2, 3, 3, 4, 4, 5], [2, 3, 3, 4]$. f. $[a, a, [b], [b], [b, b], [a, [b]]], [a, a]$.


27. b. $x \in A \cup B$ iff $x \in A$ or $x \in B$ iff $x \in B$ or $x \in A$ iff $x \in B \cup A$. Therefore, we have $A \cup B = B \cup A$. d. $x \in A \cup (B \cup C)$ iff $x \in A$ or $x \in B \cup C$ iff $x \in A$ or $x \in B$ or $x \in C$ iff $x \in A \cup B$ or $x \in C$ iff $x \in (A \cup B) \cup C$. Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$.

28. b. $x \in A \cap B$ iff $x \in A$ and $x \in B$ iff $x \in B$ and $x \in A$ iff $x \in B \cap A$. Therefore, $A \cap B = B \cap A$. d. $x \in A \cap A$ iff $x \in A$ and $x \in A$ iff $x \in A$. Therefore, $A \cap A = A$.

30. $x \in A \cup (B \cap C)$ iff $x \in A$ or $(x \in B$ and $x \in C)$ iff $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ iff $x \in (A \cup B) \cap (A \cup C)$. So $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

31. b. This is entirely similar to part (a). Use (1.4e) for the second proof.

33. b. Proof: $x \in A \setminus (B \cap A)$ iff $x \in A$ and $x \notin B \cap A$ iff $x \in A$ and $(x \notin B$ or $x \notin A)$ iff $x \in A$ and $x \notin B$ iff $x \in A \setminus B$. d. Counterexample: $A = \{a\}$.

34. b. Note that $U \ominus = U$ and $U \setminus U = \emptyset$. d. $x \in (A \cup B)'$ iff $x \notin A \cup B$ iff $x \notin A$ and $x \notin B$ iff $x \in A' \cap B'$. Therefore, $(A \cup B)' = A' \cap B'$. f. $A \cap (A' \cup B) = (A \cap A') \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$.

35. $A = \{a, \{a, A, b\}, b\} = \{a, \{a, \{a, A, b\}, b\}, b\} = \{a, \{a, \{a, \ldots, b\}, b\}, b\}$.

Section 1.3

2. b. $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)\}$. d. $\{(a), (b), (c)\}$. f. $A \times B$.

4. b. head($\langle a, b, c \rangle$) = $a$ and tail($\langle a, b, c \rangle$) = $\langle b, c \rangle$.
  d. head($\langle\langle a, b\rangle, \langle a, c\rangle\rangle$) = $\langle a, b \rangle$ and tail($\langle\langle a, b\rangle, \langle a, c\rangle\rangle$) = $\langle\langle a, b\rangle\rangle$.

5. b. $\langle 36, 36, 2, 3, 4, 6, 9, 12, 18, 36 \rangle$.

6. $\langle a, a, a \rangle, \langle a, a, b \rangle, \langle a, a, b \rangle, \langle a, b, a \rangle, \langle a, b, b \rangle, \langle b, a, a \rangle, \langle b, a, b \rangle, \langle b, b, a \rangle, \langle b, b, b \rangle, \langle b \rangle \rangle, \langle b \rangle \rangle, \langle b \rangle \rangle, \langle b \rangle \rangle$.

7. $aa, ab, ac, ba, bb, bc, ca, cb, cc$.

8. b. $ML = \{bba, bbaabb, bbab, ab, ababb, abb, aabb\}$. d. $L^1 = L = \{A, abb, b\}$.
9. b. \( L = \{ \Lambda, ba \} \). d. \( L = \{ \Lambda, b, ab \} \). f. \( L = \{ b, ab \} \).
10. b. \( x \in L \) or \( x = s(u_i \ldots u_n) \), where \( s \in L \) and \( u_k \in M \).
   d. \( x = \Lambda \) or \( x = u_i \ldots u_n \), where \( u_k \in L \cap M \).
   f. \( x = \Lambda \) or \( x \in L \) or \( x = (s_1u_1 \ldots (s_nu_n) \) where \( s_k \in L \) and \( u_k \in M \).
11. b. \( \{ a, b \}^* - \{ b \}^* \) is the set of strings over \( \{ a, b \} \) that contain at least one \( a \).
12. b. \( \{(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\} \).
   d. \( S = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\} \).
13. b. \( \{ x \mid (x, y, \text{Pacific Ocean}) \in \text{Borders for some } y \} \).
   d. \( \{(x, z) \mid (x, y, z) \in \text{Borders for some } y \} \).
14. b. \( 1(1)(5^3) - 1(1)(3^3) = 98 \). d. \( 5^6 - 4^6 - 4^6 + 3^6 = 8,162 \).
16. b. \( (x, y) \in (A - B) \times C \) iff \( x \in A \) and \( x \notin B \) and \( y \in C \) iff \( (x, y) \in (A \times C) \) and \( (x, y) \notin (B \times C) \) iff \( (x, y) \in (A \times C) - (B \times C) \). So \( (A - B) \times C = (A \times C) - (B \times C) \).
17. b. The statement is true because concatenation needs two strings and there is no string in \( \emptyset \). d. The proof is similar to that of part (c).
18. b. By definition we have \( L^* = \{ \Lambda \} \cup L^+ \). Since \( \Lambda \in L \) iff \( \Lambda \in L^* \), it follows that \( \Lambda \in L \) iff \( L^+ = L^* \).
   d. We’ll prove the equality \( (L^* M^*)* = (L \cup M)^* \). Since \( L \subseteq L \cup M \) and \( M \subseteq L \cup M \) it follows that \( L^* \subseteq (L \cup M)^* \) and \( M^* \subseteq (L \cup M)^* \). Taking the product, we have \( L^* M^* \subseteq (L \cup M)^* \). But \( (L \cup M)^* (L \cup M)^* = (L \cup M)^* \) by part (c). Therefore, \( L^* M^* \subseteq (L \cup M)^* \). Now take the closure of both sides and apply part (c) again to obtain \( (L^* M^*)^* \subseteq ((L \cup M)^*)^* = (L \cup M)^* \). Thus we have the containment \( (L^* M^*)^* \subseteq (L \cup M)^* \). For the other containment we have \( L = L \cdot \{ \Lambda \} \subseteq L^* M^* \) and \( M = \{ \Lambda \} M \subseteq L^* M^* \). Therefore, \( L \cup M \subseteq L^* M^* \), which implies \( (L \cup M)^* \subseteq (L^* M^*)^* \). Thus \( (L^* M^*)^* = (L \cup M)^* \). A similar argument can be used to show the other equality \( (L^* \cup M^*)^* = (L \cup M)^* \).
19. b. Notice that the two sets \( \{ \{ x \}, \{ x, y \} \} \) and \( \{ \{ u \}, \{ u, v \} \} \) are equal if and only if \( \{ x \} = \{ u \} \) and \( \{ x, y \} = \{ u, v \} \), and these latter equalities are true if and only if \( x = u \) and \( y = v \).
20. b. The statement \( (a, b, c) = (d, e, f) \) means \( ((a, b), c) = ((d, e), f) \). By (18b) the latter equality is true if and only if \( (a, b) = (d, e) \) and \( c = d \). Using (18b) again yields the equivalent statement \( a = d, b = e, \) and \( c = f \).
21. b. If \( a \) is an \( m \) by \( n \) matrix, then the address polynomial for the row-major location of \( a[i, j] \) is \( B + nM(i - 1) + M(j - 1) \).
Section 1.4


3. The chromatic number is 4. For example, look at Tennessee.

4. b.

6. a. One answer is $f c e j i b d a g h$. b. One answer is $f j e c d g h b a i$.

9. The tree has the following form:

11.

13. Two spanning trees are $\{\{a, b\}, \{b, g\}, \{g, c\}, \{g, f\}, \{c, d\}\}$ and $\{\{a, b\}, \{a, f\}, \{f, e\}, \{f, g\}, \{g, c\}\}$.
15. a. One possible answer: 

![Tree Diagram]

b. One possible answer: 

![Tree Diagram]

17. a. A height zero ternary tree has one node, the root, which is also the leaf. Each node has a maximum of three children. So a height one ternary tree has a maximum of three leaves, a height two ternary tree has a maximum of nine leaves, and so on.

b. There is one node at the root and a maximum of three children for each node. So the maximum number of nodes for a ternary tree of height 0 is 1, which we can write as \((3^0 + 1 - 1)/2\). For height 1 the maximum number of nodes is \(1 + 3 = 4\), which we can write as \((3^{1+1} - 1)/2\). For height 2 the maximum number of nodes is \(1 + 3 + 9 = 13\), which we can write as \((3^{2+1} - 1)/2\). In general, the maximum number of nodes for height \(n\) is \(1 + 3 + 3^2 + \ldots + 3^n = (3^{n+1} - 1)/2\). To see this, multiply both sides by 2 but use \((3 - 1)\) on the left side.

19. The graph is connected, and all vertices have even degree.

Chapter 2

Section 2.1

1. b. There are three functions of type \( \{a\} \rightarrow \{1, 2, 3\} \); one maps \(a\) to 1; another maps \(a\) to 2; and the third maps \(a\) to 3.

d. There are eight functions of type \( \{a, b, c\} \rightarrow \{1, 2\} \); two are constant functions; two functions maps \(a\) to one number and \(\{b, c\}\) to the other; two functions maps \(b\) to one number and \(\{a, c\}\) to the other; and two functions maps \(c\) to one number and \(\{a, b\}\) to the other.

2. b. \(\{x \mid x = 4k + 1 \text{ where } k \in \mathbb{N}\}\). d. \(\emptyset\). f. \(\mathbb{N}\).

3. b. -4. d. 5.

4. For example, let \(n = 3\) and \(x = \frac{1}{2}\).

6. b. 7.

8. b. 9. d. 9.

9. b. \(f(\{0, 3\}) = \{0\}\). d. \(f(\{3, 5\}) = \{0, 4\}\). f. \(f(\mathbb{N}_6) = \{0, 2, 4\}\).

10. b. ceiling\((x) = \) if \(x \leq 0\) then trunc\((x)\) else if \(x = \) trunc\((x)\) then \(x\) else trunc\((x + 1)\).

12. Yes. For example, \(5.3 \mod 2.1 = 5.3 - (2.1)\text{floor}(5.3/2.1) = 5.3 - (2.1)(2) = 1.1\). The
range of the function $f$ defined by $f(x) = x \mod 2.5$ is the half open interval $[0, 2.5)$.

13. b. 13. d. 25. f. 2.

14. b. $a^k (n/b^k) \log_b a = a^k a^{\log_b (n/b^k)} = a^k a^{\log_b n - \log_b b^k} = a^k a^{\log_b n - k \log_b b} = a^k a^{\log_b n - k} = a^{\log_b n^k} = a^{\log_b n^k} = n^{\log_b a}.$

15. b. $\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x)$.

16. b. $\emptyset.$ d. $\{0, 3\}.$

18. b. For any real number $x$ there is an integer $k$ such that $k - 1 < x \leq k$ and $k = \lfloor x \rfloor.$ Now add an integer $n$ to each term of the inequality to obtain $k + n - 1 < x + n \leq k + n.$ Since $k + n$ is an integer, the inequality tells us that $\lfloor x + n \rfloor = k + n.$ Since $k = \lfloor x \rfloor,$ we obtain the desired result that that $\lfloor x + n \rfloor = \lfloor x \rfloor + n.$

20. b. Similar to part (a). d. Similar to part (c).

21. b. Similar to part (a).

22. b. The equation holds if and only if $(x - 1)/3 \leq n \leq x/3$ for some integer $n$. Multiply the inequality by 3 to obtain $x - 1 \leq 3n \leq x.$ Add 1 to each term in the inequality to obtain the inequality $x \leq 3n + 1 \leq x + 1.$ Therefore, $3n \leq x \leq 3n + 1.$ It follows that for any integer $n$, if $3n \leq x \leq 3n + 1$, then the equation holds.

23. b. $p(x) = 1 + x^2 + x^3.$ c. $p(x) = 2 + 5x + x^2 + 10x^5.$

24. b. Let $r = \log_{b}(xy)$ and $s = \log_{b} x$ and $t = \log_{b} y.$ We need to show $r = s + t.$ The log definition tells us that $xy = b^r$, $x = b^s$ and $y = b^t$. Therefore, $b^r = xy = b^s b^t = b^{s+t}$, which implies $r = s + t.$ d. Let $r = \log_{b}(x/y)$ and $s = \log_{b} x$ and $t = \log_{b} y.$ Then proceed as in part (b) to show $r = s - t.$ f. Apply $\log_{b}$ to both sides and then use (2.5c) to obtain equal expressions.

25. b. Let $g = \gcd(a, b)$ and let $h = \gcd(b, a - bq).$ Since $g|a$ and $g|b,$ it follows that $g|(a - bq).$ Therefore, $g$ is a common divisor of $b$ and $a - bq,$ which implies $g \leq h.$ Since $hl$ and $h[(a - bq)$ there are integers $x$ and $y$ such that $b = hx$ and $a - bq = hy.$ Solve for $a$ to obtain $a = hy + bq = hy + hxq = h(y + xq).$ Therefore, $h$ is a common divisor of $a$ and $b,$ which implies $h \leq g.$ Since $g \leq h$ and $h \leq g,$ we have $g = h.$

d. Since $d = \gcd(x, y),$ it follows from (2.2c) that we can write $d = mx + ny$ for some integers $m$ and $n.$ Since $c | x$ and $c | y,$ we can write $x = cs$ and $y = ct$ for some integers $s$ and $t.$ Now we can substitute for $x$ and $y$ in the first equation to obtain

$$d = mx + ny = mcs + nct = c(mn + nt).$$

Therefore, $c | d.$
26. The inequality becomes \( 0 \leq r < -b. \) Since \( q = (a - r)/b, \) it follows (after some inequalities) that \( (a/b) \leq q < (a/b) + 1. \) Since \( q \) is an integer, we have \( q = \text{ceiling}(a/b). \)

27. **b.** If \( x \in f(E \cap F), \) then there is some element \( a \in E \cap F \) such that \( x = f(a). \) Therefore, \( x = f(a) \in f(E) \cap f(F). \)

28. **b.** We’ll prove both containments at once: \( x \in f^{-1}(G \cap H) \) iff \( f(x) \in G \cap H \) iff \( f(x) \in G \) and \( f(x) \in H \) iff \( x \in f^{-1}(G) \) and \( x \in f^{-1}(H) \) iff \( x \in f^{-1}(G) \cap f^{-1}(H). \)

**d.** If \( x \in f(f^{-1}(G)), \) then \( x = f(y) \) for some element \( y \in f^{-1}(G), \) which implies that \( f(y) \in G. \) Therefore, \( x \in G, \) which proves the containment.

29. **b.** By (2.4a) it suffices to show that \( n \) divides \( (xy) \) – \( ((x \mod n)(y \mod n)) \). Using the definition of mod we can write

\[
x \mod n = x - nq_1 \quad \text{and} \quad y \mod n = y - nq_2.
\]

So we have

\[
(xy) - ((x \mod n)(y \mod n)) = xy - ((x - nq_1)(y - nq_2)) \\
\quad = xy - (xy - nq_1x - nq_2y + n^2q_1q_2) \\
\quad = n(q_1x - q_2y + nq_1q_2).
\]

So \( n \) divides \( (xy) - ((x \mod n)(y \mod n)) \). The result follows from part (a).

32. Since \( x \mid n \) and \( y \mid n \), we can write \( n = xs = yt \) for some integers \( s \) and \( t. \) Since \( x \mid m \) and \( y \mid m, \) we can write \( m = xu = yv \) for some integers \( u \) and \( v. \) Now, by the division algorithm we can write \( n = mq + r, \) where \( 0 \leq r < m. \) To show \( m \mid n \) we must show that \( r = 0. \) Assume, by way of contradiction, that \( r > 0. \) Then \( r = n - mq \) and we can substitute for \( m \) and \( n \) to obtain the following equations.

\[
r = n - mq = xs - xuq = x(s - uq),
\]

\[
r = n - mq = yt - yvq = y(t - vq).
\]

These equations tell us that \( x \mid r \) and \( y \mid r. \) This is a contradiction to \( m \) being the smallest number such that \( x \mid m \) and \( y \mid m. \) Therefore, \( m \mid n. \)

34. Divide the inequality \( 0 < x - n < 1 \) by \( p \) to obtain \( 0 < (x - n)/p < 1/p. \) the inequality \( 0 \leq r/p \leq (p - 1)/p \) implies that \( \lfloor r/p \rfloor = 0. \) Add the last two inequalities to get \( 0 < r/p + (n - x)/p < (p - 1)/p + 1/p = 1. \) Therefore, \( \lfloor r/p \rfloor + (x - n)/p = 0. \) So \( \lfloor x/p \rfloor = q. \) Now evaluate \( \lfloor x/p \rfloor = \lfloor q + r/p \rfloor = q + \lfloor r/p \rfloor = q + 0 = q. \) So the equation holds. The proof for ceilings is similar.

35. **b.** The proof is similar to part (a) except that \( x \) can occur in one of the three intervals \( n < x < n + 1/3, \) \( n + 1/3 < x < n + 2/3, \) and \( n + 2/3 < x < n + 1. \)

**c.** If \( p \) is a positive integer, then

\[
\lfloor x \rfloor + \lfloor x + 1/p \rfloor + \lfloor x + 1/p \rfloor + \ldots + \lfloor x + (p - 1)/p \rfloor = \lfloor px \rfloor.
\]

The proof generalizes the proofs of parts (a) and (b). The main point is to consider an interval containing \( x \) such as \( n + k/p \leq x < n + (k + 1)/p. \) Multiply the inequality by \( p \) to
obtain $pn + k \leq px < pn + k + 1$. So $\lfloor px \rfloor = pn + k$. Also conclude from the first inequality that the $p - k$ numbers $x, x + 1/p, \ldots, x + (p - k - 1)/p$ are all less than $n + 1$. So they all have floor equal to $n$. The $k$ numbers $x + (p - k)/p, \ldots, x + (p - 1)/p$ are all greater than or equal to $n + 1$ and less than $n + 2$. So they have floor equal to $n + 1$. So the sum of the floors is $(p - k)n + k(n + 1) = pn - kn + kn + k = pn + k$. So the equation holds.

Section 2.2

1. b. 5. f. $\langle (0, 0), (1, 1), (2, 2), (3, 3) \rangle$.
2. b. $f(g(x)) = \lfloor (2x + 1)/2 \rfloor$, $f(f(x)) = 2\lfloor x/2 \rfloor + 1$, $f(2) = 2$, $f(f(2)) = 3$.
3. b. $g(f(x), f(y))$. d. $g(f(x), g(f(y), f(z)))$.
4. b. $2^6 < x \leq 2^7$.
5. b. $\langle 0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4 \rangle$.
6. b. $f(n) = \text{map}(\ast, \text{dist}(2, \text{seq}(n)))$.
7. b. $f(n, k) = \text{map}(\ast, \text{dist}(k, \text{seq}(n)))$. d. $f(n) = \text{map}(\ast, \text{dist}(n, \text{seq}(n)))$.
8. f. $f(n) = \text{map}(\ast, \text{dist}(1, \text{even}(n)))$, where even($n$) is a solution to Exercise 5.
9. h. $f(g, \langle x_1, x_2, \ldots, x_n \rangle) = \text{pairs}(\langle x_1, x_2, \ldots, x_n \rangle, \text{map}(g, \langle x_1, x_2, \ldots, x_n \rangle))$.
11. tail(dist$(x, \text{seq}(n))$) = tail(dist$(x, \langle 0, \ldots, n \rangle)$) = dist$(x, \text{tail}(\langle 0, \ldots, n \rangle))$ = dist$(x, \text{tail}(\text{seq}(n)))$.

Section 2.3

2. b. $f : A \rightarrow C$, where $f(a) = 1$, $f(b) = 2$, $f(c) = 2$. d. $f : A \rightarrow C$, where every element maps to 2.
3. b. Nine functions, six injections, no surjections, no bijections, and three with none of the properties.
4. b. If $f(x) = f(y)$, then $x + 1 = y + 1$, which upon subtracting by 1 yields $x = y$. So $f$ is injective. $f$ is not surjective because no $x$ maps to 0.

d. If $y \in \mathbb{N}$, then $f(2^y - 1) = \text{ceiling}(\log_2(2^y - 1)) = \text{ceiling}(\log_2 2^y) = \text{ceiling}(y \log_2 2) = \text{ceiling}(y) = y$. So $f$ is surjective. $f$ is not injective because, for example, $f(2) = f(3) = 2$.
6. b. Let $f(x) = f(y)$. Then $1/(x + 1) = 1/(y + 1)$, which says that $x = y$. So $f$ is injective. To show that $f$ is surjective, let $y \in (0, 1)$ and find a number $x > 0$ such that $f(x) = y$. Solving the equation $f(x) = y$ for $x$ yields $x = (1 - y)/y$ and we have $f((1 - y)/y) = y$. Since $0 < y < 1$ it follows that $(1 - y)/y > 0$. Thus $f$ is surjective. Therefore, $f$ is bijective.

d. Let $f(x) = f(y)$. Then $1/(2x - 1) + 1 = 1/(2y - 1) + 1$, which by elementary algebra implies that $x = y$. Therefore, $f$ is an injection. To show $f$ is surjective, let $y < 0$ and then
find some \( x \in (0, 1/2) \) such that \( f(x) = y \). Solve the equation \( f(x) = y \) to get \( x = y/(2y - 2) \). It follows that \( f(y/(2y - 2)) = y \), and since \( y < 0 \), it follows that \( 0 < y/(2y - 2) < 1/2 \). To see this, we can obtain a contradiction if \( y/(2y - 2) < 0 \) or if \( y/(2y - 2) > 1/2 \). So \( f \) is surjective. Therefore, \( f \) is bijective.

7. b. 37. d. \( 17 = 4^2 + 1 \).

8. b. The numbers in each of the three sets \{1, 6\}, \{2, 5\}, and \{3, 4\} add up to 7. Since the union of these three sets is the given set, it follows that any four numbers from the given set will contain two numbers in one of these three sets. So the sum will be 7.

9. b. Bijective. \( f^{-1}(x) = 3x \mod 5 \). The fixed point is 0. d. Not bijective because \( \gcd(3, 6) \neq 1 \). The fixed points are 2 and 5.

f. Bijective and \( f^{-1}(x) = (5x + 9) \mod 12 \). There are no fixed points.

10. b. For each \( a \neq 1 \) from part (a) we have \( \gcd(a - 1, 26) = 2 \). So if 2 does not divide \( b \), then there will be no fixed points. i.e., \( b \) can be any odd number between 0 and 26.

11. b. one, eight, two, four, six, five, three, nine, seven.

12. b. Wednesday, Monday, Friday, Tuesday, Saturday, Thursday, Sunday.


14. \( f(x) = (5x + 2) \mod 12 \).

15. b. Let \( f \) and \( g \) be surjective and let \( z \in C \). Since \( g \) is surjective, there is an element \( y \in B \) such that \( z = g(y) \). Since \( g \) is surjective, there is an element \( x \in A \) such that \( y = f(x) \). Therefore, \( z = g(y) = g(f(x)) = (g \circ f)(x) \), so it follows that \( g \circ f \) is surjective.

17. b. Suppose \( g \circ f \) is injective. Suppose also that \( f(x) = f(y) \). Apply \( g \) to both sides of the equation to obtain \( g \circ f(x) = g \circ f(y) \). So it follows that \( x = y \) because \( g \circ f \) is injective. Therefore, \( f \) is injective if \( g \circ f \) is injective.

18. b. Let \( g \) or \( h \) be injective and suppose \( f(x) = f(y) \). Then \((g(x), h(x)) = (g(y), h(y))\). Tuple equality implies that \( g(x) = g(y) \) and \( h(x) = h(y) \). Since one of \( g \) and \( h \) is injective it must be the case that \( x = y \). Therefore, \( f \) is injective. Now let \( A = \{1, 2, 3\} \), \( B = \{4, 5\} \), and \( C = \{6, 7\} \). Defined \( g(1) = 4 \), \( g(2) = 4 \), and \( g(3) = 5 \). Define \( h(1) = 6 \), \( h(2) = 7 \), and \( h(3) = 7 \). Then both \( g \) and \( h \) are not injective. But \( f \) is injective: \( f(1) = (4, 6) \), \( f(2) = (4, 7) \), and \( f(3) = (5, 7) \).
20. The statement that “$f$ has no fixed points if and only if $\gcd(a - 1, n)$ does not divide $b$” is equivalent to “$f$ has a fixed point if and only if $\gcd(a - 1, n)$ divides $b$.” We’ll prove the latter statement with the following sequence of iff statements.

$f$ has a fixed point
iff $f(x) = x$ for some $x$
iff $(ax + b) \mod n = x$ for some $x$
iff $(ax + b) \mod n = x \mod n$ for some $x$
iff $(b - (1 - a)x) \mod n = 0$ for some $x$
iff $\gcd(1 - a, n)$ divides $b$ (By Exercise 19)
iff $\gcd(a - 1, n)$ divides $b$.

Section 2.4

1. b. Let $A$ be the set. The smallest number in $A$ is $0^2 = 0$ and the largest number in $A$ is $(22)^2 = 484$. So the function $f : \{0, 1, ..., 23\} \to A$ defined by mapping $f(x) = x^2$ is a bijection. Therefore, $|A| = 23$.

2 b. Let $N$ be the set of negative integers. For example, the function $f : \mathbb{N} \to N$ defined $f(n) = -(n + 1)$ is a bijection. So $N$ is countable.

d. Let $L$ be the set of set of lists over \{a\} that have even length. So $L = \{\langle \rangle, \langle a, a \rangle, \langle aaaa \rangle, \ldots \}$. The mapping from $\mathbb{N}$ to $S$ that maps each $n$ to a list of length $2n$ is a bijection. So $L$ is countable.

f. Let $O$ be the set of odd integers. For example, the function $f : \mathbb{N} \to O$ defined $f(x) = x + 1$ when $x$ is even and $f(x) = -x$ when $x$ is odd is a bijection. So $O$ is countable.

3. b. Let $L$ be the set of all lists over \{a, b\}. For each natural number $n$ let $S_n$ be the set of all lists over \{a, b\} that have length $n$. For example, $S_0 = \{\langle \rangle\}$, $S_1 = \{\langle a \rangle, \langle b \rangle\}$ and , $S_2 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle\}$. It follows that $L = S_0 \cup S_1 \cup \ldots \cup S_n \cup \ldots$. Since each set $S_n$ is finite, hence countable, it follows from (2.11) that the union is countable.

d. For each $n$ let $S_n^* = \{(x, y, z) \mid x + y + z = n\}$. It follows that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} = S_0 \cup S_1 \cup \ldots \cup S_n \cup \ldots$. Each $S_n^*$ is finite, hence countable, so by (2.11) the union is countable.

4. b. Let $g(n) = a$ if $f(n, n) = b$, and let $g(n) = b$ if $f(n, n) \neq b$. Then the sequence $(g(0), g(1), \ldots, g(n), \ldots)$ is not in the given set.

6. Suppose, by way of contradiction, that $A - B$ is countable. Then we have $A = B \cup (A - B)$, which is the union of two countable sets. So by (2.11) we conclude that the union is countable, and thus also $A$ is countable. This contradicts the assumption that $A$ is uncountable.
7. b. Let \( A \) be a countable set and let \( f \) be an arbitrary function with domain \( A \). Since \( A \) is countable, there is a surjection from \( \mathbb{N} \) to \( A \) by part (b). The function \( f \) is a surjection from \( A \) to the image \( f(A) \). Since a composition of surjections is an surjection, we have an surjection from \( \mathbb{N} \) to \( f(A) \). Therefore, \( |f(A)| \leq |\mathbb{N}|. \)

8. b. \( A^* = A_0 \cup A_1 \cup \ldots \cup A_n \cup \ldots \) Each of these sets is countable by part (a). Therefore, \( A^* \) is countable by (2.11).

10. If \( y \in S \), then the definition of \( S \) tells us that \( y \notin S_y \). But \( S_y = S \), so the assumption that \( y \in S \) implies that \( y \notin S \), a contradiction. If \( y \notin S \), then the definition of \( S \) tells us that \( y \in S_y \). But, as before we have \( S_y = S \), so the assumption that \( y \notin S \) implies that \( y \in S \), a contradiction. Therefore, no bijections can exist between \( A \) and \( \text{power}(A) \), which gives us the desired result.

**Chapter 3**

**Section 3.1**

1. b. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

2. b. Basis: 0 \( \in S \); Induction: If \( x \in S \), then \( x + 2 \in S \).

d. Basis: \(-1 \in S \); Induction: If \( x \in S \), then \( x \pm 3 \in S \).

f. Basis: 1 \( \in S \); Induction: If \( x \in S \), then \( 2x + 1 \in S \).

3. b. Basis: 3, 4 \( \in S \). Induction: If \( x \in S \) and \( x \) is odd then \( 2x - 1 \in S \); if \( x \in S \) and \( x \) is even then \( 2x \in S \).

4. b. Basis: 2, 3 \( \in S \); Induction: If \( x \in S \), then \( x + 4 \in S \).

d. Basis: 5 \( \in S \); Induction: if \( x \in S \), then \( x + 7 \in S \).

6. b. Basis: \( \Lambda \in S \). Induction: If \( x \in S \), then \( axa \in S \).

d. Basis: \( \Lambda \in S \). Induction: If \( x \in S \), then \( ax, xb \in S \).

f. Basis: \( a \in S \). Induction: If \( x \in S \), then \( ax, xb \in S \).

h. Basis: \( ab \in S \). Induction: If \( x \in S \), then \( ax, xb \in S \).

j. Basis: \( \Lambda, b \in S \). Induction: Let \( x \in S \). If \( x = ay \) for some \( y \), then \( axa \in S \) else \( bbx \in S \).

7. b. Basis: \( a, b \in S \). Induction: If \( x \in S \), then \( axa, bxb \in S \).

d. Basis: 0, 1 \( \in S \). Induction: If \( x \in S \), then \( 0x, 1x \in S \).

9. b. \( \langle 1 \rangle, \langle 2, 1 \rangle, \langle 4, 2, 1 \rangle, \langle 8, 4, 2, 1 \rangle, \langle 16, 8, 4, 2, 1 \rangle \).

10. b. Basis: \( \langle 1 \rangle \in S \). Induction: If \( L \in S \), then \( \text{head}(L) + 1 \) \( :: L \in S \).

b. Basis: \( \langle \rangle \in S \). Induction: If \( L \in S \), then \( a :: a :: L \in S \).

f. Basis: \( \langle \rangle \in S \). Induction: If \( L \in S \) and \( a, b \in A \), then \( a :: b :: L \in S \).

h. Basis: \( \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle \in S \). Induction: If \( L \in S \) and \( a, b \in \{0, 1, 2\} \), then \( a :: b :: L \in S \).
11. **b. Basis**: \( \langle 1 \rangle \in S \). **Induction**: If \( L \in S \), then \( \text{consR}(L, \text{headR}(L) + 1) \in S \).

12. **Basis**: \( \langle \rangle, \langle a \rangle, \langle b \rangle \in S \). **Induction**: If \( x \in S \) and \( x \neq \langle \rangle \), then if \( \text{head}(x) = a \) then \( b :: x \in S \), else \( a :: x \in S \).

14. **Basis**: \( \text{tree}(\langle \rangle, n, \langle \rangle) \in B \) for all \( n \in \mathbb{N} \). **Induction**: If \( L, R \in B \), then \( \text{tree}(L, +, R), \text{tree}(L, -, R) \in B \).

16. **b.** \( \{(x, y) \mid x \neq y \text{ and } y \neq 0\} \).

17. **b. Basis**: \( (0, 0) \in S \). **Induction**: If \( (x, y) \in S \), then \( (x + 2, y), (x + 2, y + 1) \in S \).

18. **b. Basis**: \( (a, \langle \rangle) \in S \) for all \( a \in A \). **Induction**: If \( (x, L) \in S \) and \( a \in A \), then \( (x, a :: L) \in S \).

**d. Basis**: \( (0, 0, 0) \in S \). **Induction**: If \( (x, y, z) \in S \) then \( (x + 1, y, z), (x, y + 1, z), (x, y, z + 1) \in S \).

21. Let \( S \) be a set satisfying the hypothesis. Let \( S_0 \) be the set of basis elements. For each natural number \( n > 0 \) let \( S_n \) denote the set of elements that can be constructed using \( n \) constructor rules from step two. \( S \) is the union of the sets \( S_n \), each of which is a countable set. Therefore, \( S \) is countable.

**Section 3.2**

2. \( \text{length}(\langle r, s, t, u \rangle) = 1 + \text{length}(\langle s, t, u \rangle) = 1 + 1 + \text{length}(\langle t, u \rangle) = 1 + 1 + 1 + 1 + \text{length}(\langle \rangle) = 1 + 1 + 1 + 0 = 4 \).

4. **b.** \( f(0) = 0 \)

   \( f(n) = f(n - 1) + \text{floor}(n/2) \).

**d.** \( f(0) = 0 \)

   \( f(n) = f(n - 1) + n \bmod (n + 2) \).

**f.** \( f(0, k) = 0 \)

   \( f(n, k) = f(n - 1, k) + n + k \).

5. **b.** \( f(\Lambda) = \Lambda; \ f(ax) = af(x)a; \ f(bx) = bf(x)b \).

**d.** \( f(x, y) = \text{if } x = \Lambda \text{ and } y = \Lambda \text{ then true } \)

   else if \( x = as \) and \( y = at \) or \( x = bs \) and \( y = bt \) then \( f(s, t) \)

   else false.

6. **b.** \( f(0) = \langle 0 \rangle \) and \( f(z) = \langle z + 1 \rangle \) if \( z \geq 0 \) then \( z :: f(z - 1) \) else \( z :: f(z + 1) \).

**d.** \( f(x, \langle \rangle) = 0 \) and \( f(x, h :: t) = h + xf(x, t) \).

**f.** \( f(a, \langle \rangle) = \langle \rangle \) and \( f(a, h :: t) = (h + a) :: f(a, t) \).

**h.** \( f(g, \langle \rangle) = \langle \rangle \) and \( f(g, h :: t) = (h, g(h)) :: f(g, t) \).

**j.** \( f(g, h, \langle \rangle) = \langle \rangle \) and \( f(g, h, k :: t) = (g(k), h(k)) :: f(g, h, t) \).
7. b. \( f(0) = \langle 0 \rangle \),
\( f(n) = \text{cat}(f(n - 1), \langle 2n \rangle) \)

d. \( f(n, 0) = \langle n \rangle \),
\( f(n, k) = \text{cat}(f(n, k - 1), \langle n + k \rangle) \).
f. \( f(g, 0) = \langle (0, g(0)) \rangle \),
\( f(g, n) = \text{cat}(f(g, n - 1), \langle (n, g(n)) \rangle) \).

9. \( \text{eq}(L, M) = \begin{cases} \text{true} & \text{if } L = M = \langle \rangle \\ \text{false} & \text{otherwise} \end{cases} \)
   
   a. If \( L \neq \langle \rangle \) and \( M \neq \langle \rangle \) then false else eq(tail(L), tail(M))

10. b. \( \text{front}(\langle x \rangle) = \langle \rangle \),
\( \text{front}(\text{cons}(h, t)) = \text{cons}(h, \text{front}(t)) \).

11. \( \text{pal}(\langle \rangle) = \text{true} \),
\( \text{pal}(\langle x \rangle) = \text{true} \),
\( \text{pal}(\text{cons}(h, t)) = \begin{cases} \text{true} & \text{if } h = \text{last}(t) \text{ then } \text{pal}(\text{front}(t)) \text{ else } \text{false} \end{cases} \).

13. Preorder: \(+a\cdot b+de\). Inorder: \(a+b\cdot d\). Postorder: \( abde+\cdot \).

14. b. \( \text{Post}(T) = \begin{cases} \langle \rangle & \text{if } T \neq \langle \rangle \\ \text{Post}(\text{left}(T)); \text{In}(\text{right}(T)); \text{print}(\text{root}(T)) & \text{else} \end{cases} \)

15. b. Equational form:
\( \text{inOrd}(\langle \rangle) = \langle \rangle \)
\( \text{inOrd}(\text{tree}(L, r, R)) = \text{cat}(\text{inOrd}(L), r :: \text{inOrd}(R)) \).

If-then form:
\( \text{inOrd}(T) = \begin{cases} \langle \rangle & \text{if } T = \langle \rangle \\ \text{cat}(\text{inOrd}(\text{left}(T)), r :: \text{inOrd}(\text{right}(T))) & \text{else} \end{cases} \)

16. b. \( f(\langle \rangle) = -1 \),
\( f(\langle L, r, R \rangle) = 1 + \text{max}(f(L), f(R)) \).

d. \( f(\langle \langle \rangle, x, \langle \rangle \rangle) = x \),
\( f(\langle L, r, R \rangle) = \begin{cases} r & \text{if } r > \text{max}(f(L), f(R)) \text{ then } r \text{ else } \text{max}(f(L), f(R)) \end{cases} \).

17. a. The tree has \( + \) as its root with children \( * \) and \( f \). The children of \( * \) are \( a \) and \( b \), and
the children of \( f \) are \( c \), \( d \), and \( e \).

b. \( \text{post}(\langle \rangle) = \langle \rangle \),
\( \text{post}(\text{op :: operands}) = \text{cat}(g(\text{operands}), \langle \text{op} \rangle) \).
\( g(\langle \rangle) = \langle \rangle \),
\( g(\langle h :: t \rangle) = \text{cat}(\text{post}(h), \text{post}(t)) \).
An if-then-else definition can be written as
\( \text{post}(\text{op :: operands}) = \begin{cases} \langle \text{op} \rangle & \text{if operands} = \langle \rangle \\ \text{cat}(\text{cat}(\text{post}(\text{head}(\text{operands})), \text{post}(\text{tail}(\text{operands}))), \langle \text{op} \rangle) & \text{else} \end{cases} \).
18. b. \( \text{sel}(k, a, \langle \rangle) = \langle \rangle \),  
\( \text{sel}(k, a, x :: t) = \text{if } x_k = a \text{ then } a :: \text{sel}(k, a, t) \text{ else } \text{sel}(k, a, t). \)

19. b. \( \text{isSubset}(K, L) = \begin{cases} \text{true} & \text{if } K = \langle \rangle \text{ then } \\ \text{if } \text{isMember}(\text{head}(K), L) \text{ then } \\ \text{isSubset}(\text{tail}(K), L) \\ \text{else } \text{false.} \end{cases} \)

d. \( \text{union}(K, L) = \begin{cases} L & \text{if } K = \langle \rangle \text{ then } \\ \text{head}(K) :: \text{union}(\text{tail}(K), \text{removeAll}(\text{head}(K), L)). \end{cases} \)

f. \( \text{difference}(K, L) = \begin{cases} \langle \rangle & \text{if } K = \langle \rangle \text{ then } \\ \langle \rangle & \text{else if } L = \langle \rangle \text{ then } \\ \text{if } \text{isMember}(\text{head}(K), L) \text{ then } \\ \text{difference}(\text{tail}(K), L) \\ \text{else } \\ \text{head}(K) :: \text{difference}(\text{tail}(K), L). \end{cases} \)

21. \( f(0) = 0, f(1) = 1, \text{ and } f(n + 2) = f(n + 1) + f(n) + \text{fib}(n + 1)\text{fib}(n). \)

23. b. 2, 2, 2.  
d. 1, 1.5, 1.4166... .  
f. 5, 3.4, 3.0235... .

24. b. \( \text{Diff}(n, m, s) = \text{get}(n, s) - \text{get}(m, s). \)

d. \( \text{Add}(a :: s, b :: t) = (a + b) :: \text{Add}(s, t). \)

f. \( \text{Map}(f, a :: s) = f(a) :: \text{Map}(f, s). \)

25. b. \( \text{tail}(\text{Primes}) = \text{tail}(\text{sieve(ints(2)))} = \text{tail}(\text{sieve(2 :: ints(3)))} = \text{tail}(\text{remove(2, ints(3)))} = \text{remove(2, ints(3))).} \)

Section 3.3

1. b. \( S \Rightarrow DS \Rightarrow 7S \Rightarrow 7DS \Rightarrow 78S \Rightarrow 78DS \Rightarrow 780S \Rightarrow 780D \Rightarrow 7801. \)

2. b. Leftmost: \( S \Rightarrow S[S] \Rightarrow [S[S]] \Rightarrow [[S]] \Rightarrow [[[ ]]]. \)

Rightmost: \( S \Rightarrow S[S] \Rightarrow S[S[S]] \Rightarrow S[[S]] \Rightarrow [[[ ]]]. \)

d. Leftmost: \( S \Rightarrow S[S] \Rightarrow [S] \Rightarrow [S[S]] \Rightarrow [S[S][S]] \Rightarrow [[S][S]] \Rightarrow [[[ ]] [S]]. \)

Rightmost: \( S \Rightarrow S[S][S] \Rightarrow S[[S][S]] \Rightarrow S[[S][S]] \Rightarrow [[[ ]] [[[ ]]]. \)

3. b. \( S \rightarrow a \mid bS. \)

d. \( S \rightarrow bAb \text{ and } A \rightarrow A \mid aA. \)

f. \( S \rightarrow ab \mid aSb. \)

h. \( S \rightarrow b \mid aSc. \)

j. \( S \rightarrow \Lambda \mid aaS. \)
4. b. $S \rightarrow AbC$ and $A \rightarrow \Lambda \mid aA$ and $C \rightarrow \Lambda \mid cC$.
d. $S \rightarrow AB$ and $A \rightarrow \Lambda \mid aA$ and $B \rightarrow b \mid bB$.

5. b. $S \rightarrow a \mid b \mid c \mid aSa \mid bSb \mid cSc$.
d. $S \rightarrow T \mid U$ and $T \rightarrow b \mid aTc$ and $U \rightarrow B \mid A$
and $B \rightarrow \Lambda \mid bB$ and $A \rightarrow \Lambda \mid aA$.

6. b. $E \rightarrow B0$, and $B \rightarrow \Lambda \mid B0 \mid B1$.

7. b. $S \rightarrow S - T \mid T$, $T \rightarrow (S) \mid D$, where $D$ denotes a decimal numeral.

8. b. $S \Rightarrow f(S) \Rightarrow f(g(S)) \Rightarrow f(g(f(S))) \Rightarrow f(g(f(f(S))))$.

9. b. $S \Rightarrow f(S) \Rightarrow f(g(S)) \Rightarrow f(g(x, S)) \Rightarrow f(g(x, f(S))) \Rightarrow f(g(x, f(b)))$.

10. $S \rightarrow S + T \mid T$, $T \rightarrow T \cdot D \mid D$, where $D$ is a decimal number.

11. b. The string $ab$ has two parse trees.
d. The string $aba$ has two parse trees.
f. The string $bb$ has two parse trees.

12. $S \rightarrow \Lambda \mid aSbS \mid bSaS$.

13. b. $S \rightarrow AB$ and $A \rightarrow \Lambda \mid Aa$ and $B \rightarrow \Lambda \mid bB$.
d. $S \rightarrow AbA$ and $A \rightarrow \Lambda \mid aA$.
f. $S \rightarrow b \mid bS$.

14. b. $S \rightarrow AcAB \mid cAB \mid AcB \mid cB$, $A \rightarrow Aa \mid a$, $B \rightarrow Bb \mid b$.

15. b. Basis: $a$, $abc \in L(G)$. Induction: If $w \in L(G)$ and $w$ has the form $axc$, then put $abxc \in L(G)$.

Chapter 4

Section 4.1


2. b. All. d. Reflexive, antisymmetric, transitive.

3. b. $A \times A$ is reflexive, symmetric, and transitive because it contains all possible pairs. If $A = \{a\}$, then $A \times A = \{(a, a)\}$, which is also antisymmetric.

4. b. $\{(a, b), (b, a)\}$. d. $\{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$. f. $\emptyset$.

5. b. isAuntOf. d. isFirstCousinOf.
7. b. Let \( R = \{(a, b), (b, b), (b, c), (c, a)\} \). Then \( R \) is anti-symmetric. But \( R^2 = \{(a, b), (a, c), (b, a), (b, c), (c, b)\} \), which is not anti-symmetric.

8. b. \([(x, y) \mid x < y - 2]\).

9. b. \([(x, y) \mid x \neq 0 \text{ and } y \neq 0]\).

10. \( R^2 = R \). To see this, notice that the condition that \( x + y \) is even means that \( x \) and \( y \) are both even or both odd. So \( (x, y) \in R^2 \) iff there is an integer \( z \) such that \( (x, z) \in R \) and \( (z, y) \in R \) iff \( x + z \) is even and \( z + y \) is even iff \( x \) and \( y \) are both even or both odd iff \( (x, y) \in R \).

12. b. \([(a, b), (b, a)]\).

13. b. \([(a, b), (a, c), (b, c)]\).

14. \( R^2 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ and } y \text{ are both even or both odd}\} \) and \( R^3 = R \). So \( t(R) = R^2 \cup R \). Since the condition that \( x \) and \( y \) are both even or both odd means the same as \( x + y \) is even, we have

\[
\begin{align*}
t(R) &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is odd}\} \cup \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is even}\} \\
&= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x + y \text{ is odd or even}\} \\
&= \mathbb{Z} \times \mathbb{Z}.
\end{align*}
\]

15. b. isDescendantOf.

18. There are 12 distinct path matrices.

19. Let \( M \) be the adjacency matrix for \( R \). b. Check to see if \( M_{ij} = M_{ji} \) for all \( i \) and \( j \). d. Set \( M_{ii} = 1 \) for all \( i \).

20. b. Let \( R \) be symmetric and let \( a R^2 b \). Then \( a R x \) and \( x R b \) for some \( x \). Since \( R \) is symmetric it follows that \( b R x \) and \( x R a \). Therefore, \( b R^2 a \). Thus \( R^2 \) is symmetric.

22. b. \((x, y) \in R \circ (S \cup T)\) iff \((x, w) \in R\) and \((w, y) \in S \cup T\) for some \( w \) iff \((x, w) \in R\) and either \((w, y) \in S\) or \((w, y) \in T\) for some \( w \) iff either \((x, y) \in R \circ S\) or \((x, y) \in R \circ T\) iff \((x, y) \in R \circ S \cup R \circ T\).

23. \((x, y) \in E \circ R\) iff \((x, x) \in E\) and \((x, y) \in R\) iff \((x, y) \in R\). Thus \( E \circ R = R \). A similar argument shows \( R \circ E = R \).

24. b. Suppose \( R \) is symmetric. If \((a, b) \in r(R)\), then either \( a = b \) or \((a, b) \in R\). In either case, \((b, a) \in r(R)\). Therefore, \( r(R) \) is symmetric. If \((a, b) \in t(R)\), then there is a sequence of elements \( a = x_1, x_2, \ldots, x_n = b \) such that \((x_i, x_{i+1}) \in R\) for \( 1 \leq i < n \). Since \( R \) is symmetric we also have \((x_{i+1}, x_i) \in R\) for \( 1 \leq i < n \), which implies that \((b, a) \in t(R)\). Therefore, \( t(R) \) is symmetric.
25. b. Letting $E$ be the equality relation on $A$, we have the following series of equations:
\[ sr(R) = s(R \cup E) = (R \cup E) \cup (R \cup E)^c = R \cup E \cup R^c \cup E^c = R \cup R^c \cup E = r(R \cup R^c) = rs(R). \]

26 b. If $R$ is asymmetric, then $(x \, R \, y \, \text{and} \, y \, R \, x)$ is always false. So the statement, “$x \, R \, y$ and $y \, R \, x$ implies $x = y$,” is vacuously true. So $R$ satisfies the definition of antisymmetric.

Section 4.2

1. b. Any string has the same letters as itself. So $\sim$ is reflexive. If $s$ and $t$ have the same occurrences of letters, then so to $t$ and $s$. So $\sim$ is symmetric. If $s$ and $t$ have the same occurrences of letters and $t$ and $u$ do also, then $s$ and $u$ have the same occurrences of letters. So $\sim$ is transitive.

d. $x - x = 0$, which is an integer for all rational $x$. So $\sim$ is reflexive. If $x - y$ is an integer, then $y - x$ is an integer too. So $\sim$ is symmetric. Let $x - y$ and $y - z$ be integers. Since adding two integers yields another integer, we obtain $x - y + y - z = x - z$ is an integer. So $\sim$ is transitive.

d. $x - x = 0$, which is an integer for all rational $x$. So $\sim$ is reflexive. If $x - y$ is an integer, then $y - x$ is an integer too. So $\sim$ is symmetric. Let $x - y$ and $y - z$ be integers. Since adding two integers yields another integer, we obtain $x - y + y - z = x - z$ is an integer. So $\sim$ is transitive.

2. b. The relation is not symmetric. For example, $4/2$ is an integer but $2/4$ is not an integer.

d. $R$ is not transitive. For example, $(2, 6) \in R$ and $(6, 12) \in R$, but $(2, 12) \notin R$.

3. b. $[n] = \{n\}$ for each $n \in \mathbb{N}$.

d. $[3n] = \{3n, 3n + 1, 3n + 2\}$ for each $n \in \mathbb{N}$.

f. $[kn] = \{kn, kn + 1, kn + 2, ..., kn + k - 1\}$ for each $n \in \mathbb{N}$.

4. b. The classes are the sets $\{x \in \mathbb{R} | k \leq x < k + 1\}$ for each $k \in \mathbb{Z}$.

5. b. Four classes $[0], [1], [2], [3]$, where $[n] = \{4k + n \mid k \in \mathbb{N}\}$.

6. b. Six classes $\{\text{rot}\}, \{\text{roto, root}\}, \{\text{tot}\}, \{\text{toot, toto, otto}\} \{\text{too}\}$, and $\{\text{to}\}$.

7. b. The weight is 7. One answer is $\{\{a, b\}, \{b, c\}, \{c, d\}, \{c, g\}, \{g, f\}, \{f, e\}\}$.

8. Let $x \in S$. Then the hypothesis says $x \, R \, y$ for some $y \in S$. Since $R$ is symmetric, it follows that $y \, R \, x$. So $x \, R \, y$ and $y \, R \, x$. Since $R$ is transitive, it follows that $x \, R \, x$. Therefore, $R$ is reflexive.

10. $[x] = \{y \mid (x, y) \in E \cap F\} = \{y \mid (x, y) \in E \text{ and } (x, y) \in F\} = \{y \mid (x, y) \in E\} \cap \{(x, y) \in F\} = [x]_E \cap [x]_F$. 

12. Call the procedure “reverseLinks” and let \( i \) be its argument. Then we can define it recursively as follows:

\[
\textbf{procedure} \ \text{reverseLinks}(i) \\
\text{\quad if } p(i) \neq 0 \text{ then} \\
\text{\quad \quad reverseLinks}(p(i)); \\
\text{\quad \quad } p(p(i)) := i; \\
\text{\quad \quad } p(i) := 0; \\
\text{\quad fi}
\]

Section 4.3


3. d.

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{a} \\
\text{b} \\
\end{array}
\]

4.

\[
(f(x) + g(x))(f(x)g(x))
\]

The poset is not a lattice because, for example, \( \{f(x), g(x)\} \) has no lub.

6. Because there are no infinite descending chains of distinct elements from a finite set.

7. b. No tree has fewer than zero leaves. Therefore, every descending sequence of trees is finite if the order is by the number of leaves.
8. One: (4, 3), (4, 2), (4, 1), (4, 0), (3, 0), (2, 0), (1, 0), (0, 0).
Two: (4, 3), (3, 3), (2, 3), (1, 3), (0, 3), (0, 2), (0, 1), (0, 0).

10. One possible answer is 1, 2, 4, 3, 5, 6, 7.

12. Suppose the hypothesis of the anti-symmetric property holds for two elements x and y: x R y and y R x. Then we must conclude that x R x holds. But this contradicts the fact that R is irreflexive. Thus the hypothesis of the anti-symmetric property is always false, which makes anti-symmetry vacuously true.

14. b. Yes. d. No. For example, 2|4 but f(2) does not divide f(4).

Section 4.4

2. b. The equation is true for n = 1. So assume that the equation is true for n, and prove that it’s true for n + 1. Starting on the left-hand side, we get

\[ 5 + 9 + 11 + \cdots + [2(n + 1) + 3] = [5 + 9 + 11 + \cdots + (2n + 3)] + [2(n + 1) + 3] \]
\[ = (n^2 + 4n) + [2(n + 1) + 3] \]
\[ = n^2 + 4n + 2n + 2 + 3 \]
\[ = (n + 1)^2 + 4(n + 1). \]

d. The equation is true for n = 1. So assume that the equation is true for n, and prove that it’s true for n + 1. Starting on the left-hand side, we get

\[ 2 + 6 + 10 + \cdots + (4n - 2) + [4(n + 1) - 2] = (2 + 6 + 10 + \cdots + (4n - 2)) + [4(n + 1) - 2] \]
\[ = 2n^2 + 4(n + 1) - 2 \]
\[ = 2n^2 + 4n + 2 \]
\[ = (2n + 2)(n + 1) \]
\[ = 2(n + 1)^2. \]

f. The equation is true for n = 1. So assume that the equation is true for n, and prove that it’s true for n + 1. Starting on the left-hand side, we get

\[ 2 + 8 + \cdots + (n + 1)2^{n + 1} = (2 + 8 + \cdots n2^n) + (n + 1)2^{n + 1} \]
\[ = 2^{n + 1}(n - 1) + 2 + (n + 1)2^{n + 1} \]
\[ = 2^{n + 1}(n - 1 + n + 1) + 2 \]
\[ = 2^{n + 2} n + 2 \]
\[ = 2^{(n + 1) + 1}((n + 1) - 1) + 2. \]
h. The equation is true for \( n = 1 \). So assume that the equation is true for \( n \), and prove that it’s true for \( n + 1 \). Starting on the left-hand side, we get

\[
2 + 6 + 12 + \cdots + (n + 1)(n + 2) = [2 + 6 + 12 + \cdots + n(n + 1)] + (n + 1)(n + 2)
\]

\[
= \frac{n(n + 1)(n + 2)}{3} + (n + 1)(n + 2)
\]

\[
= \frac{(n + 1)(n + 2)(n + 3)}{3}.
\]

j. The equation is true if \( n = 1 \). So assume the equation is true for \( n \) and prove it’s true for \( n + 1 \). Starting on the left hand side we get

\[
(1 + 2 + \cdots + n + (n + 1))^2 = [(1 + 2 + \cdots + n) + (n + 1)]^2
\]

\[
= (1 + 2 + \cdots + n)^2 + 2(1 + 2 + \cdots + n)(n + 1) + (n + 1)^2
\]

\[
= 1^3 + 2^3 + \cdots + n^3 + 2(1 + 2 + \cdots + n)(n + 1) + (n + 1)^2
\]

\[
= 1^3 + 2^3 + \cdots + n^3 + n(n + 1)(n + 1) + (n + 1)^2
\]

\[
= 1^3 + 2^3 + \cdots + n^3 + (n + 1)^2 + (n + 1)^2
\]

\[
= 1^3 + 2^3 + \cdots + n^3 + (n + 1)^3.
\]

3. b. For \( n = 1 \) the equation becomes \( 0 - 1 = -1 \). Assume that the equation is true for \( n \). Then the case for \( n + 1 \) goes as follows:

\[
F_n F_{n+2} - F^2_{n+1} = F_n (F_{n+1} + F_n) - F^2_{n+1}
\]

\[
= F_n F_{n+1} + F^2_n - F^2_{n+1}
\]

\[
= (F_n - F_{n+1})F_{n+1} + F^2_n
\]

\[
= -F_{n+1}F_{n+1} + F^2_n
\]

\[
= -(F_{n+1} - F_n^2)
\]

\[
= -(1)^n
\]

\[
= (-1)^{n+1}.
\]

d. Since \( m|n \) it follows that \( n = mk \) for some \( k \). We’ll show that \( F_m \mid F_{mk} \) for all natural numbers \( k \geq 1 \). If \( k = 1 \) then \( n = m \), so \( F_m = F_n \) and thus \( F_m \mid F_n \). Assume that \( F_m \mid F_{mk} \) and prove that \( F_m \mid F_{m(k+1)} \). We have the following representation, using part (c).

\[
F_{m(k+1)} = F_{mk+m}
\]

\[
= F_{mk+1}F_m + F_{mk}F_{m+1}.
\]

Since \( F_m \mid F_{mk} \) and \( F_m \mid F_m \), it follows that \( F_m \) divides the right side of the equation and thus also \( F_m \mid F_{m(k+1)} \).
4. b. For \( n = 1 \) the equation becomes \( 1 = 0 + 1 \). Assume the equation is true for all \( k < n \). Then the case for \( n \) goes as follows:

\[
L_n = L_{n-1} + L_{n-2} = (F_{n-2} + F_n) + (F_{n-3} + F_{n-1}) = F_{n-1} + F_{n+1}.
\]

6. The equation has the form

\[
\sum_{i=0}^{n} (n^2 + i) = \sum_{i=0}^{n-1} (n^2 + n + 1 + i).
\]

It’s easy to evaluate these sums. For example, the left expression can be evaluated as follows:

\[
\sum_{i=0}^{n} (n^2 + i) = n^2(n + 1) + \sum_{i=0}^{n} i = n^3 + n^2 + \frac{n(n + 1)}{2}.
\]

Similarly, the right side of the original equation evaluates to

\[
n^3 + n^2 + n + \frac{(n - 1)n}{2}.
\]

The resulting two expressions are easily seen to be equal by direct transformation or by induction.

8. \( \text{power}(\emptyset) = \{\emptyset\} \). So a finite set with 0 elements has \( 2^0 \) subsets. Now let \( A \) be a set with \( |A| = n > 0 \), and assume that the statement is true for any set with fewer than \( n \) elements. We can write \( A = \{x\} \cup B \), where \( |B| = n - 1 \). So we can write \( \text{power}(A) \) as the union of two disjoint sets: \( \text{power}(A) = \text{power}(B) \cup \{\{x\} \cup C \mid C \in \text{power}(B)\} \). Since \( x \notin B \), these two sets have the same cardinality, which by induction is \( 2^{n-1} \). In other words, we have \( |\{\{x\} \cup C \mid C \in \text{power}(B)\}| = \text{power}(B)| = 2^{n-1} \). Therefore, \( |\text{power}(A)| = \text{power}(B)| + |\{\{x\} \cup C \mid C \in \text{power}(B)\}| = 2^{n-1} + 2^{n-1} = 2^n \). Therefore, any finite set with \( n \) elements has \( 2^n \) subsets.

10. b. Let \( T \) be a binary tree. We know that an empty tree has no leaves. Since \( h(\langle \rangle) = 0 \), we know the function is correct when \( T = \langle \rangle \). We also know that a single node tree has one leaf. Since \( h(\text{tree}(\langle \rangle, x, \langle \rangle)) = 1 \), we know the function is correct for a single node tree. For the induction part we need a well-founded ordering on binary trees. For example, let \( t \prec s \) mean \( t \) is a subtree of \( s \). Now assume \( T \) is a nonempty binary tree with two or more nodes, and also assume the function is correct for all subtrees of \( T \). So \( T = \text{tree}(L, x, R) \) where one of \( L \) and \( R \) is nonempty. We know that the number of leaves in \( T \) is equal to the number of leaves in \( L \) plus those in \( R \). The function \( h \), when given argument \( T \), returns \( h(L) + h(R) \). Since \( L \) and \( R \) are subtrees of \( T \), it follows by assumption that \( h(L) \) and \( h(R) \) represent the number of leaves in \( L \) and \( R \), respectively. Therefore, \( h(T) \) is the number of leaves in \( T \).
11. b. Similar to the proof of part (a).

12. b. We’ll use the same well-founded ordering on lists as part (a). Certainly the result of inserting an element \( x \) in the empty list is just \( \langle x \rangle \), which is sorted. Thus the base case is correct, since \( \text{insert}(x, \langle \rangle) = \langle x \rangle \). For the induction case, assume that \( \text{insert}(x, L) \) is sorted for all sorted lists \( L \) having length \( n \), and show \( \text{insert}(x, y :: L) \) is sorted, where \( y :: L \) is sorted. If \( x \leq y \), then \( x :: y :: L \) is sorted. So in this case, \( \text{insert}(x, y :: L) = x :: y :: L \), which says that \( \text{insert}(x, y :: L) \) is defined. If \( x > y \), then \( \text{insert}(x, y :: L) = y :: \text{insert}(x, L) \). The list \( \text{insert}(x, L) \) is sorted by the induction hypothesis. Therefore, \( \text{insert}(x, y :: L) \) is sorted because \( x > y \).

14. First we must show that \( f(\langle \rangle) \) is a binary search tree. Since \( f(\langle \rangle) = \langle \rangle \) and the empty tree is a trivial binary search tree, the basis case is true. Next, we let \( x :: L \) be an arbitrary list and assume that \( f(L) \) is a binary search tree. Then we must show that \( f(x :: L) \) is a binary search tree. Using the definition of \( f \), we obtain \( f(x :: L) = \text{insert}(x, f(L)) \). Since, by assumption, \( f(L) \) is a binary search tree, it follows that \( \text{insert}(x, f(L)) \) is also a binary search tree (remember we are assuming that the insert function is correct). Therefore, \( f(x :: L) \) is a binary search tree. It follows from (4.29) that \( f(M) \) is a binary search tree for all lists \( M \). QED.

16. Let \( P(a, L) \) mean “\( \text{removeAll}(a, L) \) contains no occurrences of \( a \).” The definition gives \( \text{removeAll}(a, \langle \rangle) = \langle \rangle \). So \( P(a, \langle \rangle) \) is true for any \( a \). This proves the basis case. Now assume that \( L \neq \langle \rangle \) and assume that \( P(a, K) \) is true for all lists \( K < L \). Show that \( P(a, L) \) is true. Since there are two else clauses to the definition, we have two cases. For the first case, assume that \( a = \text{head}(L) \). In this case we have \( \text{removeAll}(a, L) = \text{removeAll}(a, \text{tail}(L)) \). The induction assumption implies that \( P(a, \text{tail}(L)) \) is true. Therefore, \( P(a, L) \) is true when \( a = \text{head}(L) \). Now assume that \( a \neq \text{head}(L) \). Then the definition gives

\[
\text{removeAll}(a, L) = \text{head}(L) :: \text{removeAll}(a, \text{tail}(L)).
\]

The induction assumption says that \( \text{removeAll}(a, \text{tail}(L)) \) is true. Since \( a \neq \text{head}(L) \), it follows that \( \text{head}(L) :: \text{removeAll}(a, \text{tail}(L)) \) has no occurrences of \( a \). Therefore, \( P(a, L) \) is true if \( a \neq \text{head}(L) \). It follows from (4.29) that \( P(a, L) \) is true for all elements \( a \) and all lists \( L \). QED.

18. Let \( \prec \) denote the lexicographic ordering on \( \mathbb{N} \times \mathbb{N} \). Then \( \mathbb{N} \times \mathbb{N} \) is a well-ordered set, hence well-founded with least element \( (0, 0) \). We’ll use (4.29) to prove that \( f(x, y) \) is defined (i.e., halts) for all \( (x, y) \in \mathbb{N} \times \mathbb{N} \). First we have \( f(0, 0) = 0 + 1 = 1 \). So \( f(0, 0) \) is defined. Thus Step 1 of (4.29) is done. Now to Step 2. Assume that \( (x, y) \in \mathbb{N} \times \mathbb{N} \), and assume that \( f(x', y') \) is defined for all \( (x', y') \) such that \( (x', y') \prec (x, y) \). To finish Step 2, we must show that \( f(x, y) \) is defined. The definition of \( f(x, y) \) gives us three possibilities:
1. If \( x = 0 \), then \( f(x, y) = y + 1 \). Thus \( f(x, y) \) is defined.

2. If \( x \neq 0 \) and \( y = 0 \), then \( f(x, y) = f(x - 1, 1) \). Since \( (x - 1, 1) \prec (x, y) \), our assumption says that \( f(x - 1, 1) \) is defined. Therefore, \( f(x, y) \) is defined.

3. If \( x \neq 0 \) and \( y \neq 0 \), then \( f(x, y) = f(x - 1, f(x, y - 1)) \). First notice that we have \( (x, y - 1) \prec (x, y) \). So our assumption says that \( f(x, y - 1) \) is defined. Thus the pair \( (x - 1, f(x, y - 1)) \) is a valid element of \( \mathbb{N} \times \mathbb{N} \). Now, since we have \( (x - 1, f(x, y - 1)) \prec (x, y) \) our assumption again applies to say that \( f(x - 1, f(x, y - 1)) \) is defined. Therefore, \( f(x, y) \) is defined.

So Steps 1 and 2 of (4.29) have been accomplished for the statement “\( f(x, y) \) is defined.” Therefore, \( f(x, y) \) is defined for all natural numbers \( x \) and \( y \). QED.

19. b. If \( L = \langle \rangle \) then both sides of the equation are false. Now assume that the equation is true for \( L \) and all elements \( a \neq b \). Then prove that the equation is true for \( c :: L \). In other words, show that

\[
isMember(a, \text{removeAll}(b, c :: L)) = isMember(a, c :: L).
\]

If \( b = c \), then the left hand side of the equation becomes (let \( r \) stand for \( \text{removeAll} \))

\[
isMember(a, r(c, c :: L)) = isMember(a, r(c, L)) \quad \text{(def of \( r \))}
\]
\[
= isMember(a, L) \quad \text{(induction)}
\]
\[
= isMember(a, c :: L) \quad \text{(def of member)}.
\]

Therefore, the equation is true when \( b = c \). Now assume that \( b \neq c \). Starting with the left-hand side of the equation we obtain

\[
isMember(a, r(b, c :: L)) = isMember(a, c :: r(b, L)) \quad \text{(def of \( r \))}
\]
\[
= \text{if } a = c \text{ then True}
\]
\[
\quad \text{else isMember}(a, r(b, L)) \quad \text{(def of member)}.
\]

Now the right-hand side of the equation can be written

\[
isMember(a, c :: L) = \text{if } a = c \text{ then True}
\]
\[
\quad \text{else isMember}(a, L) \quad \text{(def of member)}.
\]

The induction assumption says \( \text{isMember}(a, r(b, L)) = \text{member}(a, L) \). Therefore, the two conditionals above are equal, which finishes the proof.

21. a. The equation is true when \( L = \langle \rangle \). So let \( L = b :: M \) and assume that \( r(x, g(K)) = g(r(x, K)) \) for all elements \( x \) and all lists \( K \) with fewer elements that \( L \). Split the proof in two parts. First consider the case \( a = b \). In this case, starting on the left side we get

\[
\begin{align*}
r(a, g(L)) &= r(a, g(a :: M)) = r(a, a :: g(r(a, M))) = r(a, g(r(a, M))) \\
&= r(a, r(a, g(M))) = r(a, g(M)) = g(r(a, M)) = g(r(a, a :: M)) = g(r(a, L)).
\end{align*}
\]
Next consider the case \( a \neq b \). Here we start with the left side to get

\[
\begin{align*}
    r(a, g(L)) &= r(a, g(b :: M)) \\
                 &= r(a, b :: g(r(b, M))) \quad \text{(def of } g) \\
                 &= b :: r(a, g(r(b, M))) \quad (a \neq b) \\
                 &= b :: g(r(a, r(b, M))) \quad \text{(induction, since } r(b, M) < L) \\
                 &= b :: g(r(b, r(a, M))) \quad \text{(problem 13)} \\
                 &= g(b :: r(a, M)) \quad \text{(def of } g) \\
                 &= g(r(a, b :: M)) \quad (a \neq b) \\
                 &= g(r(a, L)).
\end{align*}
\]

\[\text{b.}\] It’s clear that \( f(\langle \rangle) = g(\langle \rangle) \). Now let \( L = a :: M \) and assume \( f(K) = g(K) \) for all \( K \) with fewer elements than \( L \). Now notice that

\[
f(L) = f(a :: M) = a :: r(a, f(M)) \quad \text{and} \quad g(L) = g(a :: M) = a :: g(r(a, M)).
\]

We will be done if we can show that \( r(a, f(M)) = g(r(a, M)) \). But the induction assumption allows us to write \( f(M) = g(M) \). Therefore, we will be done if we can show that \( r(a, g(M)) = g(r(a, M)) \). This is part (a). Thus \( f(L) = g(L) \) for all lists \( L \).

\[\text{c.}\] An appropriate name is “makeSet” since all redundant elements are removed by repeatedly applying the removeAll function.

**23.** We can use (3.15) to obtain the following inductive definition of \( L(G) \): Basis: \( a \in L(G) \). Induction: If \( x \in L(G) \), then put \( abx \in L(G) \). To prove that \( M = L(G) \) we can show that \( M \) satisfies the inductive definition for \( L(G) \) and \( M \subseteq L(G) \). Equality will then follow from (4.31). Since \( a = (ab)^0 a \), it follows that \( a \in M \). Therefore, the basis case holds for \( M \). If \( x \in M \), then \( x = (ab)^n a \) for some \( n \). It follows that \( abx = (ab)^{n+1} a \in M \). So the induction case holds for \( M \). Now use an induction proof to show \( M \subseteq L(G) \) by showing \( (ab)^n a \in L(G) \) for all \( n \in \mathbb{N} \). If \( n = 0 \) then \( (ab)^0 a = a \in L(G) \). For the induction part of the proof, assume \( (ab)^n a \in L(G) \) and prove \( (ab)^{n+1} a \in L(G) \). Since \( (ab)^n a \in L(G) \), the inductive definition tells us that \( ab(ab)^n a \in L(G) \). In other words, \( (ab)^{n+1} a \in L(G) \). Therefore, \( M \subseteq L(G) \). So (4.31) tells us that \( M = L(G) \).

**24.** \[\text{b.}\] Similar to part (a).
Chapter 5

Section 5.1

2. b. 5.
4. There are 25 possibilities. Therefore, a ternary pan balance algorithm must make at least three comparisons to solve the problem. One example is

```
1,2,3,4 5,6,7,8
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
1,6,7,8 5,9,10,11
```

Section 5.2

1. b. \(1(3^1) + 2(3^2) + 3(3^3) + 4(3^4) + 5(3^5)\).
2. b. \(\sum_{i=2}^{n+1} g(i-2)a_{i-1}x^i\). d. \(\sum_{i=3}^{n+2} g(i-3)a_{i-2}x^{i-1}\).
3. b. \(\sum_{i=0}^{n} (6i + 3) = 6\sum_{i=0}^{n} i + 3\sum_{i=0}^{n} 3 = \frac{6n(n+1)}{2} + (3)(n+1) = 3(n+1)^2\).
d. \(\sum_{i=1}^{n} i3^i = \frac{3 - (n+1)3^{n+1} + n3^{n+2}}{2^2}\).
4. b. \(n^2\). d. \(n(2n+1)\). f. \((n-1)2^{n+1} + 2\). h. \(\frac{n(n^2 - 1)}{3}\).
8. b. \(\sum_{i=0}^{k-1} 3^{2i}(1/5)^{i\log_5 3} = \sum_{i=0}^{k-1} 3^{2i}5^{i\log_5 3} = \sum_{i=0}^{k-1} 3^{2i}3^{-i} = \sum_{i=0}^{k-1} 3^i = (3^k - 1)/2\).
10. b. \(H_{2n-1} - (1/2)H_{n-1}\).
11. b. \(\sum_{k=1}^{n} \frac{k^2}{k+1} = \sum_{k=1}^{n} \left((k-1) + \frac{1}{k+1}\right) = (1/2)n(n-1) + H_{n+1} - 1\).
13. b. \((n^2 + 3n)/2\).
14. b. \(1 + n + (2 + 3 + \ldots + (n + 1)) = n + (1/2)(n + 1)(n + 2)\).
15. b. \(1 + (k + 1) + (3 + 5 + \ldots + (2k + 3))\), where \(k = \lceil n/2 \rceil - 1\), which has the value \(1 + \lceil n/2 \rceil \lceil (n/2) + 3 \rceil\).
18. b. \(\frac{n^3}{3} \leq \frac{n(n+1)(2n+1)}{6} \leq \frac{(n+1)^3}{3} - \frac{1}{3} = \frac{n(n^2 + 3n + 3)}{3}\).
20. b. Calculate the difference to obtain the following for \(n > 2\).
\[
\ln n + \frac{1}{2n} + \frac{1}{2} - \ln \left(\frac{n+1}{2}\right) - 1 = \ln \left(\frac{2n}{n+1}\right) + \frac{1}{2n} - \frac{1}{2} > \ln \left(\frac{2n}{n+1}\right) + \frac{1}{2n} > 0.
\]

Section 5.3
2. b. \(ab, ac, ba, bc, ca, cb\). d. \(\{a, b\}, \{a, c\}, \{b, c\}\).
3. b. \(P(3, 2)\). d. \(C(3, 2)\).
5. b. \(5!/(2!2!1!) = 30\). d. \(11!/(4!4!2!1!) = 34,650\).
6. b. BA. d. \(BCDA, BADC, BDAC, CADB, CDAB, DABC, DCAB, DCBA\).
8. \(C(5, 2)\cdot C(4, 2)\cdot C(6, 3) = 1200\).
9. b. Use the binomial theorem (5.9) with \(x = y = 1\) and (5.10) to obtain the result.
11. It works!
13. There are \((2n)!/(n!n!)\) total strings of length \(2n\) with \(n\) zeros and \(n\) ones. Any string that is not OK must contain a zero with an equal number of zeros and ones to its left. This collection of strings has the same cardinality as the set of all strings of length \(2n\) that contain \(n + 1\) zeros and \(n - 1\) ones. This latter set has cardinality \((2n)!/(n + 1)!(n - 1)!\). So the desired number of strings is \((2n)!/(n!n!) - (2n)!/(n + 1)!(n - 1)!\), which simplifies to \((2n)!/(n!(n + 1)!\).

Section 5.4
1. b. 0.375. d. 0.875.
2. b. 0.5. d. 0.8333....
3. b. 211/243.
4. b. 0.68. Calculation: \(1 - P(\text{no hits in four at bats}) = 1 - C(4, 0)(1/4)^0(3/4)^4\).
5. 2/11, 6/11, 3/11.
7. b. 1/3.

8. The first processor takes 0.01 second to do its job, and the second processor takes 0.005 second to do its job. Therefore, 2000 operations are processed in 0.015 second, which equals 133,333 operations per second.

9. b. The total number of possible 5-element sets is $C(49, 5)$. Suppose that the winning set of five numbers is $\{a, b, c, d, e\}$. Then there are 220 5-element sets that contain 4 of the five numbers. To see this, notice that there are 44 sets of the form $\{x, b, c, d, e\}$, where $x \in \{1, 2, ..., 49\} – \{a, b, c, d, e\}$. Similarly, there are 44 sets of the form $\{a, x, c, d, e\}$, where $x \in \{1, 2, ..., 49\} – \{a, b, c, d, e\}$ and so on for $x = c, x = d,$ and $x = e$. Therefore, the probability of winning a smaller prize is $220/C(49, 5)$.

d. $C(m, k)C(n – m, m – k)/C(n, m)$.

10. b. 25.

15. Since $A$ and $B$ are independent events, we have $P(A \cap B) = P(A)P(B)$. To show that $A$ and $B'$ are independent, notice that $A \cap B' = A – (A \cap B)$. So by the definition of the probability of an event, we have

\[
P(A \cap B') = P(A - (A \cap B)) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B').
\]

So $A$ and $B'$ are independent events. The other proofs are similar.

17. b. Solve the inequality $0.99 \leq 1 - [C(n, 0)(1/2)^n + C(n, 1)(1/2)^n]$, which can be written as $100 \leq 2^n/(n + 1)$. Trial and error yields $n = 11$.

### Section 5.5

1. b. $n^2 + n – 2$.

2. b. Let $a_n$ be the number of cons operations when $L$ has length $n$. Then $a_0 = 0$ and $a_n = 3 + a_{n-1}$, which has solution $a_n = 3n$.

4. Let $D_n$ denote the number of diagonals in an $n$-sided polygon. Then $D_3 = 0$ and $D_n = D_{n-1} + (n – 2)$. Solving the recurrence gives $D_n = n(n – 3)/2$.

6. b. $a_n = (-2)^n + (1/2)(-1/6)^n$.

7. b. $a_n = 4^n – 3^n$.

8. b. $A(x) = 4(1/(1-x)^3) - 2(1/(1-x)^2) - 2(1/(1-x)) + 2$, which yields $a_n = n^2 + n – 2$.
9. b. Starting with the left side we have:
\[
\frac{(1)(1)(3) \cdots (2n-3)}{n!} \cdot 2^n = \frac{(1)(1)(2)(3)(4) \cdots (2n-4)(2n-3)(2n-2)}{(2)(4) \cdots (2n-4)(2n-2)n!} \\
= \frac{(2n-2)!}{2^{n-1}(n-1)!n!} = \frac{2(2n-2)!}{n(n-1)!(n-1)!} = 2 \frac{2n-2}{n(n-1)}.
\]

Section 5.6

1. b. The hypotheses tell us that there are positive constants \(c, d, m_1, e, \) and \(m_2\) such that \(chl(n) \leq f(n) \leq dhl(n)\) for \(n \geq m_1\) and \(0 \leq g(n) \leq dhl(n)\) for \(n \geq m_2\). Let \(m\) be the larger of \(m_1\) and \(m_2\). By adding the two inequalities we obtain \(chl(n) \leq f(n) + g(n) \leq (d + e)hl(n)\) for \(n \geq m\). Therefore, \(f(n) + g(n) = \Theta(h(n))\).

2. b. \((1/2)(1) \leq 1 - 1/n \leq (1)(1)\) for \(n \geq 2\).

4. b. The quotient \(\log(k + n)/\log n\) approaches 1 as \(n\) approaches infinity.

5. b. If \(\varepsilon > 0\) and \(f(n) = \Omega(n^{k+\varepsilon})\), then \(|f(n)| \leq cn^{k+\varepsilon}|.\) So \(\ln|f(n)| \leq \ln cn^{k+\varepsilon}\) which goes to 0 as \(n\) goes to infinity. Therefore, \(n^{k} = o(f(n))\).

7. \(\lim_{n \to \infty} \frac{n}{n \log n} = \lim_{n \to \infty} \frac{1}{\log n} = 0\) and \(\lim_{n \to \infty} \frac{n \log n}{n^2} = \lim_{n \to \infty} \frac{\log n}{n} = 0\).

10. b. \(10! = 3,628,800;\) Stirling = 3,598,694; diff = 30,106.

11. b. \(f(n) = \Theta(n^2)\). Notice that
\[
f(n) = \log(1 \cdot 2 \cdot 2^2 \cdots 2^n) = \log(2^{1+2+\cdots+n}) \\
= (1 + 2 + \cdots + n) \log 2 = \frac{n(n+1)}{2} \log 2 = \Theta(n^2).
\]

14. b. The assumption tells us that \(|f(n)| \leq c_1|g(n)|\) for \(n \geq m_1\) and \(|g(n)| \leq c_2|h(n)|\) for \(n \geq m_2\). Therefore, \(|f(n)| \leq c_1|g(n)| \leq c_1c_2|g(n)|\) for \(n \geq \max\{m_1, m_2\}\). Thus \(f(n) = O(h(n))\).

d. Since \(f(n) = O(g(n))\), there are positive numbers \(c\) and \(m\) such that \(|f(n)| \leq c|g(n)|\) for \(n \geq m\). It follows that \(|af(n)| = |a||f(n)| \leq |a|c|g(n)| = c|g(n)|.\) So \(af(n) = O(ag(n))\).

f. The hypothesis tells us there are positive constants \(c_1, m_1, c_2,\) and \(m_2\) such that \(|f_1(n)| \leq c_1|g_1(n)|\) for \(n \geq m_1\) and \(|f_2(n)| \leq c_2|g_2(n)|\) for \(n \geq m_2\). Let \(m\) be the larger of \(m_1\) and \(m_2\). Then \(|f_1(n)f_2(n)| \leq c_1c_2|g_1(n)g_2(n)|\) for \(n \geq m\). So \(f_1(n)f_2(n) = O(g_1(n)g_2(n))\).

15. b. Let \(h(n) = O(g(n))\). Since \(f(n) = O(f(n))\), it follows that \(f(n)h(n) = O(f(n))O(g(n))\). Therefore, \(f(n)O(g(n)) = O(f(n))O(g(n))\).
d. Let \( g(n) = O(af(n)) \). Then there are positive constants \( c \) and \( m \) such that \( |g(n)| \leq c\alpha f(n) \) for \( n \geq m \). We can rewrite the inequality as \( |g(n)| \leq c\alpha f(n) = (c\alpha)f(n) \). If \( a = 0 \), then \( g(n) = 0 \) so that \( g(n) = O(f(n)) \). If \( a \neq 0 \), then \( c\alpha \) is a positive constant and the inequality gives us \( g(n) = O(f(n)) \). Therefore, \( O(af(n)) = O(f(n)) \).

f. The statement follows by induction on \( n \). If \( n = 1 \), the statement is then \( O(f(n)) = O(f(n)) \), which is true. Assume the statement is true for \( n \). Then the statement for \( n + 1 \) is also true by using part (e) as follows.

\[
\sum_{k=1}^{n+1} O(f(k)) = \sum_{k=1}^{n} O(f(k)) + O(f(n + 1)) = O\left(\sum_{k=1}^{n} f(k)\right) + O(f(n + 1)) = O\left(\sum_{k=1}^{n+1} f(k)\right).
\]

16. b. The assumption tells us that \( |f(n)| \geq c_n g(n) \) for \( n \geq n_1 \) and \( |g(n)| \geq c_n h(n) \) for \( n \geq n_2 \). Therefore, \( |f(n)| \geq c_n |g(n)| \geq c_n c_n g(n) \) for \( n \geq \max\{n_1, n_2\} \). Thus \( f(n) = \Omega(h(n)) \).

d. If \( f(n) = \Omega(g(n)) \), then there are positive constants \( c \) and \( m \) such that \( |f(n)| \geq c |g(n)| \) for \( n \geq m \). Since \( a \neq 0 \), multiply both sides of the inequality by \( a \) to obtain \( |af(n)| \geq (c|a|)g(n) \) for \( n \geq m \). Thus \( af(n) = O(g(n)) \).

f. Since \( f_1 \) and \( f_2 \) are nonnegative, there are positive constants such that \( f_1(n) \geq c_1 |g_1(n)| \) for \( n \geq m_1 \) and \( f_2(n) \geq c_2 |g_2(n)| \) for \( n \geq m_2 \). Let \( c \) be the smaller of \( c_1 \) and \( c_2 \) and let \( m \) be the larger of \( m_1 \) and \( m_2 \). Then \( f_1(n) + f_2(n) \geq c_1 |g_1(n)| + c_2 |g_2(n)| \geq c(|g_1(n)| + |g_2(n)|) \geq c|g_1(n)| + c|g_2(n)| \) for \( n \geq m \). Therefore, \( f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n)) \).

17. b. Since \( f(n) = \Theta(g(n)) \), there are positive constants \( c, d, \) and \( m \) such that \( c \leq |g(n)| \leq d |f(n)| \) for \( n \geq m \). It follows that \((1/d) |f(n)| \leq |g(n)| \leq (1/c) |f(n)| \) for \( n \geq m \). Therefore, \( g(n) = \Theta(f(n)) \).

d. Since \( f(n) = \Theta(g(n)) \), there are positive constants \( c, d, \) and \( m \) such that \( |c| \leq |g(n)| \leq |f(n)| \) for \( n \geq m \). So for any number \( a \) it follows that \( |c| |g(n)| \leq |af(n)| \leq |d| |g(n)| \). Therefore, \( af(n) = \Theta(ag(n)) \).

18. b. Let \( g(n) = \Theta(af(n)) \). Then \( \beta |g(n)| \leq |f(n)| \leq \alpha |g(n)| \) for positive constants \( \alpha \) and \( \beta \). We can rewrite this inequality as \( (\beta|a|) |f(n)| \leq |g(n)| \leq (\alpha|a|) |f(n)| \) since \( a \neq 0 \), the constants \( \beta \) and \( \alpha \) are positive. So we have \( g(n) = \Theta(f(n)) \). Therefore, \( \Theta(af(n)) = \Theta(f(n)) \).

d. The statement follows by induction on \( n \). If \( n = 1 \), the statement is \( f(n) = \Theta(g(n)) \), which is true by hypothesis. Assume the statement is true for \( n \). Then the statement for \( n + 1 \) is also true by using part (f) as follows.

\[
\sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n + 1) = \Theta\left(\sum_{k=1}^{n} g(k) + g(n + 1)\right) = \Theta\left(\sum_{k=1}^{n+1} g(k)\right).
\]

19. b. Let \( h_1(n) = o(f(n)) \) and \( h_2(n) = o(f(n)) \). Let \( \varepsilon > 0 \). Then there are positive constants \( m_1 \) and \( m_2 \) such that \( |h_1(n)| \leq (\varepsilon/2) |f(n)| \) for \( n \geq m_1 \) and \( |h_2(n)| \leq (\varepsilon/2) |f(n)| \) for \( n \geq m_2 \). Let \( m \) be the larger of \( m_1 \) and \( m_2 \). It follows that \( |h_1(n) + h_2(n)| \leq |h_1(n)| + |h_2(n)| \leq (\varepsilon/2) |f(n)| + (\varepsilon/2) |f(n)| = \varepsilon |f(n)| \) for \( n \geq m \). So \( h_1(n) + h_2(n) = o(f(n)) \). Therefore, \( o(f(n)) + o(f(n)) = o(f(n)) \).
d. Let \( h(n) = o(o(f(n))) \). Then there is a function \( g \) such that \( h(n) = o(g(n)) \) and \( g(n) = o(f(n)) \). It follows from part (a) that \( h(n) = o(f(n)) \). Therefore, \( o(o(f(n))) = o(f(n)) \).

20. b. Let \( h(n) = o(f(n)) \) and \( g(n) = O(f(n)) \). Then there are positive constants \( c \) and \( m_1 \) such that \( |g(n)| \leq cf(n) \) for \( n \geq m_1 \). For the same \( c \), (here \( c \) is an epsilon that we pick) there is a positive constant \( m_2 \) such that \( |h(n)| \leq cf(n) \) for \( n \geq m_2 \). Let \( m \) be the larger of \( m_1 \) and \( m_2 \). It follows that \( |h(n) + g(n)| \leq |h(n)| + |g(n)| \leq cf(n) + cf(n) = (2c)|f(n)| \) for \( n \geq m \). Thus \( h(n) + g(n) = O(f(n)) \). Therefore, \( o(f(n)) + O(f(n)) = O(f(n)) \).

d. Let \( f(n) = O(g(n)) \) and \( h_i(n) = \Theta(f(n)h(n)) \). Then there are positive constants \( c_1, m_1, c_2, d_2, \) and \( m_2 \) such that \( |f(n)| \leq c_1|g(n)| \) for \( n \geq m_1 \) and \( |f(n)h(n)| \leq |h_i(n)| \leq d_2|f(n)h(n)| \) for \( n \geq m_2 \). Let \( m \) be the larger of \( m_1 \) and \( m_2 \). It follows that \( |h_i(n)| \leq d_2|f(n)h(n)| \leq d_2c_1|g(n)|h(n) | \) for \( n \geq m \). Thus \( h(n) = O(g(n)h(n)) \). Therefore, \( \Theta(f(n)h(n)) = O(g(n)h(n)) \).

f. Let \( f(n) = o(g(n)) \) and \( h_i(n) = O(f(n)h(n)) \). Then there are positive constants \( c \) and \( m_1 \) such that \( |h_i(n)| \leq cf(n)h(n) \) for \( n \geq m_1 \). Let \( \varepsilon > 0 \). Since \( f(n) = o(g(n)) \) we will pick \( \varepsilon/c \) for which there is a positive constant \( m_2 \) such that \( |f(n)| \leq (\varepsilon/c)|g(n)| \) for \( n \geq m_2 \). Let \( m \) be the larger of \( m_1 \) and \( m_2 \). It follows that \( |h_i(n)| \leq cf(n)h(n) \) \( \leq c(\varepsilon/c)|g(n)|h(n) \) \( |n \geq m \). Thus \( h_i(n) = o(g(n)h(n)) \). Therefore, \( O(f(n)h(n)) = o(g(n)h(n)) \).

### Chapter 6

#### 6.2

1. b. \( ((P \lor ((\neg Q) \land R)) \rightarrow (P \lor R)) \rightarrow (\neg Q) \).

2. b. \( A \rightarrow B \lor C \rightarrow A \lor \neg B \).


6. The statement \( (\text{False} \rightarrow \text{True}) \rightarrow \text{False} \) is a contradiction while the statement \( \text{False} \rightarrow (\text{True} \rightarrow \text{False}) \) is a tautology.

7. b. If \( A = \text{True} \), then the wff is true. If \( A = \text{False} \) and \( B = \text{True} \), then the wff is false.

   d. If \( B = \text{True} \), then the wff is true. If \( A = \text{True} \), \( B = \text{False} \), and \( C = \text{True} \), then the wff is false.

   f. If \( A = B = \text{True} \), then the wff is true. If \( B = \text{True} \), and \( A = C = \text{False} \), then the wff is false.

8. b. If \( C = \text{True} \) or \( D = \text{True} \), then the consequent is true, so the statment is trivially true.

   If \( C = D = \text{False} \), then the wff becomes \( (A \lor B) \land (A \lor \text{False}) \land (B \lor \text{False}) \rightarrow \text{False} = (A \lor B) \land \neg A \land \neg B \rightarrow \text{False} \).

   If \( A = \text{True} \), then the wff becomes \( (\text{True} \lor B) \land \text{False} \land \neg B \rightarrow \text{False} \), which is equivalent to \( \text{False} \rightarrow \text{False} = \text{True} \). If \( A = \text{False} \), then the wff becomes \( (\text{False} \lor B) \land \text{True} \land \neg B \rightarrow \text{False} \), which is equivalent to \( B \land \neg B \rightarrow \text{False} = \text{True} \).
d. If $C = \text{True}$, then the wff is trivially true. If $C$ is false, then the wff becomes $(A \to (B \to \text{False})) \to ((A \to B) \to (A \to \text{False}))$

$\equiv (A \to \neg B) \to ((A \to B) \to \neg A)$. If $A$ is false then the wff is trivially true. If $A = \text{True}$, then the wff becomes

$(\text{True} \to \neg B) \to ((\text{True} \to B) \to \text{False}) \equiv \neg B \to (B \to \text{False}) \equiv \neg B \equiv \text{True}$.

f. If either $B$ or $C$ is true, then the wff is trivially true. If $B$ and $C$ are false, then the wff becomes $(A \to \text{False}) \to (\text{False} \lor A \to \text{False})$

$\equiv \neg A \to (A \to \text{False}) \equiv \neg A \equiv \text{True}$.

h. If $C$ is false, then the wff is trivially true. If $C = \text{True}$, then the wff becomes $(A \to B) \to (\neg (B \land \text{True}) \to \neg (\text{True} \land A)) \equiv (A \to B) \to (\neg B \to \neg A)$. If $A$ is false, then the wff is trivially true. If $A = \text{True}$, then the wff becomes

$(\text{True} \to B) \to (\neg B \to \text{False}) \equiv B \equiv B \equiv \text{True}$.

9. b. $(A \to C) \lor (B \to C) \equiv \neg A \lor C \land \neg B \lor C \equiv \neg A \lor \neg B \lor C \equiv \neg (A \land B) \lor C$

$\equiv A \land B \to C$.

d. $A \lor B \to C \equiv \neg (A \land B) \lor C \equiv \neg A \lor \neg B \lor A \equiv \text{True} \lor \neg B \equiv \text{True}$.

d. $A \to B \lor C \equiv \neg A \lor B \lor C \equiv \neg A \lor (\neg A \lor C) \equiv (A \to B) \lor (A \to C)$.

10. b. $A \land B \to A \equiv \neg (A \land B) \lor A \equiv \neg A \lor \neg B \lor A \equiv \text{True} \lor \neg B \equiv \text{True}$.

d. $A \to (B \to A) \equiv \neg A \lor (\neg B \lor A) \equiv \neg A \lor B \lor A \equiv \text{True} \lor \neg B \equiv \text{True}$.

f. $(A \to B) \land A \to B \equiv (\neg A \lor B) \land A \to B \equiv ((\neg A \lor B) \land A) \lor B$

$\equiv (A \land \neg B) \lor (\neg A \lor B) \equiv (A \lor \neg A \lor B) \land (\neg B \lor A \lor B)$

$\equiv (\text{True} \lor B) \land (\text{True} \lor \neg A) \equiv \text{True} \land \text{True} \equiv \text{True}$.

h. $(A \to B) \to ((A \to \neg B) \to \neg A) \equiv (A \to B) \lor (\neg (A \to \neg B) \lor \neg A)$

$\equiv (A \land \neg B) \lor ((A \land B) \land \neg A) \equiv (A \land \neg B) \lor (B \lor \neg A)$

$\equiv (A \land \neg B) \lor (A \land \neg B) \equiv \text{True}$.

11. b. $\neg P \lor \neg Q \lor P$ or $\neg P \lor P$, since the wff is a tautology.

d. $(P \land R) \lor (Q \land R)$. f. $(A \land \neg C) \lor (B \land \neg C) \lor (A \land D) \lor (B \land D)$.

12. b. $P \lor \neg P$, since the wff is a tautology. d. $(P \lor Q) \land R$.

f. $(A \lor E \lor F) \land (B \lor E \lor F)$.

13. b. Full DNF: $(P \land Q \land R) \lor (P \land Q \land \neg R) \lor (P \land \neg Q \land \neg R) \lor (\neg P \land Q \land R)$

$\lor (\neg P \land Q \land \neg R) \lor (\neg P \land \neg Q \land \neg R)$. Full CNF: $(\neg P \lor Q \lor \neg R) \land (P \lor Q \lor \neg R)$.

14. b. $(\neg Q \land P) \lor (\neg Q \land \neg P) \lor (P \land Q)$.

d. $(P \land Q \land R) \lor (P \land \neg Q \land R) \lor (\neg P \land Q \land R)$.

15. b. No full CNF, since the wff is a tautology.

d. $(\neg P \lor Q \lor R) \land (\neg P \lor Q \lor \neg R) \land (\neg P \lor \neg Q \lor R)$.

16. b. $A \lor B \equiv \neg A \rightarrow B$. Therefore, $\{\neg, \rightarrow\}$ is an adequate set because $\{\neg, \lor\}$ is adequate.

d. $\neg A \equiv \text{NAND}(A, A)$, and $A \land B \equiv \neg \text{NAND}(A, B) \equiv \text{NAND}(\text{NAND}(A, B), \text{NAND}(A, B))$. Therefore, NAND is adequate because $\{\neg, \land\}$ is an adequate set of connectives.
17. If \( f(\text{True, True}) = \text{True} \), then negation can’t be represented in terms of \( f \) because we would obtain \( \neg \text{True} = \text{True} \). Similarly, if \( f(\text{False, False}) = \text{False} \), then we would obtain \( \neg \text{False} = \text{False} \). Therefore, \( f(\text{True, True}) = \text{False} \) and \( f(\text{False, False}) = \text{True} \). Now consider the four possible cases for the remaining two values \( f(\text{True, False}) \) and \( f(\text{False, True}) \). One case gives NAND; one case gives NOR; and the other two cases give negations, which are not complete.

Section 6.3

1. b. \((A \rightarrow B) \land A \rightarrow B\).

3. b. One premise: The premise of the conditional is \((A \rightarrow B) \rightarrow C\).

5. b. 1. \( A \)  
2. \( \neg B \)  
3. \( A \land \neg B \)  
4. \( \neg B \rightarrow A \land \neg B \)  

QED

\[ 1, 2, \text{Conj} \]

5. b. 1. \( B \rightarrow C \)  
2. \( A \land B \)  
3. \( B \)  
4. \( C \)  
5. \( A \)  
6. \( A \land C \)  
7. \( (A \land B) \rightarrow (A \land C) \)  

QED

\[ 1, 4, \text{CP} \]

5. d. 1. \( B \rightarrow C \)  
2. \( A \land B \)  
3. \( B \)  
4. \( C \)  
5. \( A \land C \)  
6. \( (A \land B) \rightarrow (A \land C) \)  

QED

\[ 1, 7, \text{CP} \]

5. f. 1. \( A \lor B \rightarrow C \)  
2. \( C \rightarrow D \land E \)  
3. \( A \)  
4. \( A \lor B \)  
5. \( C \)  
6. \( D \land E \)  
7. \( D \)  
8. \( A \rightarrow D \)  

QED

\[ 1, 2, 8, \text{CP} \]
h.  1.  \( A \rightarrow (B \rightarrow C) \)  
    \( P \)

2.  \( B \)  
    \( P \) [for \( B \rightarrow (A \rightarrow C) \)]

3.  \( A \)  
    \( P \) [for \( A \rightarrow C \)]

4.  \( B \rightarrow C \)  
    1, 3, MP

5.  \( C \)  
    2, 4, MP

6.  \( A \rightarrow C \)  
    3–5, CP

7.  \( B \rightarrow (A \rightarrow C) \)  
    2, 6, CP

QED  
    1, 7, CP.

j.  1.  \( A \rightarrow C \)  
    \( P \)

2.  \( A \)  
    \( P \) [for \( A \rightarrow B \lor C \)]

3.  \( C \)  
    1, 2, MP

4.  \( B \lor C \)  
    3, Add

5.  \( A \rightarrow B \lor C \)  
    2–4, CP

QED  
    1, 5, CP.

6.  b.  1.  \( A \rightarrow B \)  
    \( P \)

2.  \( A \lor B \)  
    \( P \)

3.  \( \neg B \)  
    \( P \) [for \( B \)]

4.  \( A \)  
    2, 3, DS

5.  \( B \)  
    1, 4, MP

6.  \( \text{False} \)  
    3, 5, Contr

7.  \( B \)  
    3–6, IP

QED  
    1, 2, 7, CP.

d.  1.  \( A \rightarrow C \)  
    \( P \)

2.  \( A \)  
    \( P \) [for \( A \rightarrow (B \lor C) \)]

3.  \( \neg (B \lor C) \)  
    \( P \) [for \( B \lor C \)]

4.  \( C \)  
    1, 2, MP

5.  \( B \lor C \)  
    4, Add

6.  \( \text{False} \)  
    3, 5, Contr

7.  \( B \lor C \)  
    3–6, IP

8.  \( A \rightarrow (B \lor C) \)  
    2, 7, CP

QED  
    1, 8, CP.
f. 1. \( A \rightarrow B \)  
   \( P \)
2. \( B \rightarrow C \)  
   \( P \) [for \( (B \rightarrow C) \rightarrow (A \lor B \rightarrow C) \)]
3. \( A \lor B \)  
   \( P \) [for \( A \lor B \rightarrow C \)]
4. \( \neg C \)  
   \( P \) [for \( C \)]
5. \( \neg B \)  
   2, 4, MT
6. \( \neg A \)  
   1, 5, MT
7. \( B \)  
   3, 6, DS
8. \( \text{False} \)  
   5, 7, Contr
9. \( C \)  
   4–8, IP
10. \( A \lor B \rightarrow C \)  
    3, 9, CP
11. \( (B \rightarrow C) \rightarrow (A \lor B \rightarrow C) \)  
    2, 10, CP
QED
12. 1, 11, CP.

h. 1. \( C \rightarrow A \)  
   \( P \)
2. \( \neg C \rightarrow B \)  
   \( P \)
3. \( \neg (A \lor B) \)  
   \( P \) [for \( A \lor B \)]
4. \( \neg A \)  
   \( P \) [for \( A \)]
5. \( \neg C \)  
   1, 4, MT
6. \( B \)  
   2, 5, MP
7. \( A \lor B \)  
   6, Add
8. \( \text{False} \)  
   3, 7, Contr
9. \( A \)  
   4–8, IP
10. \( A \lor B \)  
    9, Add
11. \( \text{False} \)  
    3, 10, Contr
12. \( A \lor B \)  
    3, 9–11, IP
QED
1, 2, 12, CP.

7. For some proofs we’ll use IP in a subproof.

b. 1. \( B \rightarrow C \)  
   \( P \)
2. \( A \land B \)  
   \( P \) [for \( A \land B \rightarrow A \land C \)]
3. \( \neg (A \land C) \)  
   \( P \) [for \( A \land C \)]
4. \( B \)  
   2, Simp
5. \( C \)  
   1, 4, MP
6. \( A \)  
   2, Simp
7. \( A \land C \)  
   5, 6, Conj
8. \( \text{False} \)  
   3, 7, Contr
9. \( A \land C \)  
   3–8, IP
10. \( A \land B \rightarrow A \land C \)  
    2, 9, CP
QED
1, 10, CP.
d. 1. $A \lor B \rightarrow C$  
2. $C \rightarrow D \land E$  
3. $A$  
4. $\neg D$  
5. $A \lor B$  
6. $C$  
7. $D \land E$  
8. $D$  
9. False  
10. $D$  
11. $A \rightarrow D$  

QED  

f. 1. $A \rightarrow B$  
2. $B \rightarrow C$  
3. $A \lor B$  
4. $\neg C$  
5. $\neg B$  
6. $\neg A$  
7. $A$  
8. False  
9. $C$  
10. $A \lor B \rightarrow C$  
11. $(B \rightarrow C) \rightarrow (A \lor B \rightarrow C)$  

QED  

h. 1. $A \rightarrow C$  
2. $A \land B$  
3. $\neg C$  
4. $\neg A$  
5. $A$  
6. False  
7. $C$  
8. $A \land B \rightarrow C$  

QED
9. b. 1. \( A \rightarrow B \)  
2. \( \neg A \rightarrow C \)  
3. \( A \)  
4. \( B \)  
5. \( A \land B \)  
6. \( A \rightarrow A \land B \)  
7. \( \neg A \)  
8. \( C \)  
9. \( \neg A \land C \)  
10. \( \neg A \rightarrow \neg A \land C \)  
11. \( A \lor \neg A \)  
12. \( (A \land B) \lor (\neg A \land C) \)  

QED

10. b. Proof of \( A \land False \rightarrow False \):

1. \( A \land False \)  
2. \( False \)  

QED 1, 2, MP.

Proof of \( False \rightarrow A \land False \):

1. \( False \)  
2. \( \neg A \)  
3. \( False \)  
4. \( A \)  
5. \( A \land False \)  

QED 1, 2, 3, MP.

d. Proof of \( A \land \neg A \rightarrow False \):

1. \( A \land \neg A \)  
2. \( A \)  
3. \( \neg A \)  
4. \( False \)  

QED 1, 2, 3, MP.
Proof of False → A ∧ ¬ A:

1. False  
2. ¬ A  
3. False  
4. A
5. ¬ ¬ A
6. False  
7. ¬ A
8. A ∧ ¬ A

QED 1, 4, 7, 8, CP.

f. Proof of A ∧ (B ∧ C) → (A ∧ B) ∧ C:

1. A ∧ (B ∧ C)  
2. A
3. B ∧ C
4. B
5. C
6. A ∧ B
7. (A ∧ B) ∧ C

QED 1–7, CP.

The proof of (A ∧ B) ∧ C → A ∧ (B ∧ C) is similar.

11. b. Proof of A ∨ False → A:

1. A ∨ False  
2. ¬ A  
3. False
4. A

QED 1, 4, CP.

Proof of A → A ∨ False:

1. A
2. A ∨ False

QED 1, 2, CP.
d. Proof of \( A \lor (B \lor C) \to (A \lor B) \lor C \):

1. \( A \lor (B \lor C) \)  \( P \)
2. \( A \)  \( P \) [for \( A \to (A \lor B) \lor C \)]
3. \( A \lor B \)  2, Add
4. \( (A \lor B) \lor C \)  3, Add
5. \( A \to (A \lor B) \lor C \)  2–4, CP
6. \( B \lor C \)  \( P \) [for \( B \lor C \to (A \lor B) \lor C \)]
7. \( B \)  \( P \) [for \( B \to (A \lor B) \)]
8. \( A \lor B \)  7, Add
10. \( B \to A \lor B \)  7, 8, CP
11. \( C \to C \)  \( T \) [Example 6.11]
12. \( (A \lor B) \lor C \)  6, 10, 11, CD
13. \( B \lor C \to (A \lor B) \lor C \)  6, 10–12, CP
14. \( (A \lor B) \lor C \)  1, 5, 13, Cases
QED  1, 5, 13, 14, CP.

The proof of \( (A \lor B) \lor C \to A \lor (B \lor C) \) is similar.

12. b. Proof \( (A \to \text{False}) \to \neg A \):

1. \( A \to \text{False} \)  \( P \)
2. \( \neg \neg A \)  \( P \) [for \( \neg A \)]
3. \( A \)  2, DN
4. \( \neg A \)  1, 3, MP
5. \( \neg \neg A \)  2–4, IP
QED  1, 5, CP.

Proof of \( \neg A \to (A \to \text{False}) \):

1. \( \neg A \)  \( P \)
2. \( A \)  \( P \) [for \( A \to \text{False} \)]
3. \( \text{False} \)  1, 2, Contr
4. \( A \to \text{False} \)  2, 3, CP
QED  1, 4, CP.

d. Proof of \( \text{False} \to A \):

1. \( \text{False} \)  \( P \)
2. \( \neg A \)  \( P \) [for \( A \)]
3. \( \text{False} \land \neg A \)  1, 2, Conj
4. \( \text{False} \)  3, Simp
5. \( A \)  2, 3, IP
QED  1, 7, CP.
13. **b.** Proof of \((A \to B) \to (A \land \neg B \to \text{False})\)

1. \(A \to B\) \(\quad P\)
2. \(A \land \neg B\) \(P\) [for \(A \land \neg B \to \text{False}\)]
3. \(A\) \(\quad 2, \text{Simp}\)
4. \(B\) \(\quad 1, 3, \text{MP}\)
5. \(\neg B\) \(\quad 2, \text{Simp}\)
6. False \(\quad 4, 5, \text{Contr}\)
7. \(A \land \neg B \to \text{False}\) \(2–6, \text{CP}\)

QED \(1, 7, \text{CP}.\)

Proof of \((A \land \neg B \to \text{False}) \to (A \to B):\)

1. \(A \land \neg B \to \text{False}\) \(\quad P\)
2. \(A\) \(\quad 1, \text{Simp}\)
3. \(B\) \(\quad P\) [for \(B\)]
4. \(A \land \neg B\) \(\quad 2, 3, \text{Conj}\)
5. False \(\quad 1, 4, \text{MP}\)
6. \(B\) \(\quad 3–5, \text{IP}\)
7. \(A \to B\) \(\quad 2, 6, \text{CP}\)

QED \(1, 7, \text{CP}.\)

**d.** Proof of \(A \land (B \lor C) \to (A \lor B) \lor (A \land C):\)

1. \(A \land (B \lor C)\) \(\quad P\)
2. \(A\) \(\quad 1, \text{Simp}\)
3. \(B \lor C\) \(\quad 1, \text{Simp}\)
4. \(B\) \(\quad P\) [for \(B \to (A \land B)\)]
5. \(A \land B\) \(\quad 2, 4, \text{Conj}\)
6. \(B \to (A \land B)\) \(\quad 4, 5, \text{CP}\)
7. \(C\) \(\quad P\) [for \(C \to (A \land C)\)]
8. \(A \land C\) \(\quad 2, 7, \text{Conj}\)
9. \(C \to A \land C\) \(7, 8, \text{CP}\)
10. \((A \land B) \lor (A \land C)\) \(\quad 3, 6, 9, \text{CD}\)

QED \(1–3, 6, 9, 10, \text{CP}.\)
The proof of \((A \land B) \lor (A \land C) \rightarrow A \land (B \lor C)\):

1. \((A \land B) \lor (A \land C)\) \(P\)
2. \(A \land B\) \(P\) [for \(A \land B \rightarrow A \land (B \lor C)\)]
3. \(A\) \(2,\ \text{Simp}\)
4. \(B\) \(2,\ \text{Simp}\)
5. \(B \lor C\) \(4,\ \text{Add}\)
6. \(A \land (B \lor C)\) \(3, 5,\ \text{Conj}\)
7. \(A \land B \rightarrow A \land (B \lor C)\) \(3–6,\ \text{CP}\)
8. \(A \land C\) \(P\) [for \(A \land C \rightarrow A \land (B \lor C)\)]
9. \(A\) \(8,\ \text{Simp}\)
10. \(C\) \(8,\ \text{Simp}\)
11. \(B \lor C\) \(10,\ \text{Add}\)
12. \(A \land (B \lor C)\) \(9, 11,\ \text{Conj}\)
13. \(A \land B \rightarrow A \land (B \lor C)\) \(3–6,\ \text{CP}\)
14. \(A \land (B \lor C)\) \(1, 7, 13,\ \text{Cases}\)
\(\text{QED}\) \(1, 7, 13, 14,\ \text{CP}\).

**14. b.** Proof of \(A \lor (A \land B) \rightarrow A\):

1. \(A \lor (A \land B)\) \(P\)
2. \(\neg A\) \(P\) [for \(A\)]
3. \(A \land B\) \(1, 2,\ \text{DS}\)
4. \(A\) \(3,\ \text{Simp}\)
5. \(\text{False}\) \(2, 4,\ \text{Contr}\)
6. \(A\) \(2–5,\ \text{IP}\)
\(\text{QED}\) \(1, 6,\ \text{CP}\).

Proof of \(A \rightarrow A \lor (A \land B)\):

1. \(A\) \(P\)
2. \(A \lor (A \land B)\) \(1,\ \text{Add}\)
\(\text{QED}\) \(1, 2,\ \text{CP}\).

**d.** Proof of \(A \lor (\neg A \land B) \rightarrow A \lor B\):

1. \(A \lor (\neg A \land B)\) \(P\)
2. \(A\) \(P\) [for \(A \rightarrow A \lor B\)]
3. \(A \lor B\) \(2,\ \text{Add}\)
4. \(A \rightarrow A \lor B\) \(2, 3,\ \text{CP}\)
5. \(\neg A \land B\) \(P\) [for \(\neg A \land B \rightarrow A \lor B\)]
6. \(B\) \(5,\ \text{Simp}\)
7. \(A \lor B\) \(6,\ \text{Add}\)
8. \(\neg A \land B \rightarrow A \lor B\) \(5–7,\ \text{CP}\)
9. \(A \lor B\) \(1, 4, 8,\ \text{Cases}\)
\(\text{QED}\) \(1, 4, 8, 9,\ \text{CP}\).
Proof of $A \lor B \rightarrow A \lor (\neg A \land B)$:

1. $A \lor B$ \hspace{1cm} P
2. $A$ \hspace{1cm} P [for $A \rightarrow A \lor (\neg A \land B)$]
3. $A \lor (\neg A \land B)$ \hspace{1cm} 2, Add
4. $A \rightarrow A \lor (\neg A \land B)$ \hspace{1cm} 2, 3, CP
5. $\neg A$ \hspace{1cm} P [for $\neg A \rightarrow A \lor (\neg A \land B)$]
6. $B$ \hspace{1cm} 1, 5, DS
7. $\neg A \land B$ \hspace{1cm} 5, 6, Conj
8. $A \lor (\neg A \land B)$ \hspace{1cm} 7, Add
9. $\neg A \rightarrow A \lor (\neg A \land B)$ \hspace{1cm} 6–8, CP
10. $A \lor \neg A$ \hspace{1cm} T [Example 6.17]
11. $A \lor (\neg A \land B)$ \hspace{1cm} 4, 9, 10, Cases
QED

**Section 6.4**

3. 1. $\neg A \rightarrow \neg B$ \hspace{1cm} P
2. $(\neg A \rightarrow \neg B) \rightarrow (\neg \neg B \rightarrow \neg \neg A)$ \hspace{1cm} Axiom 4
3. $\neg \neg B \rightarrow \neg \neg A$ \hspace{1cm} 1, 2, MP
4. $B \rightarrow \neg \neg B$ \hspace{1cm} Axiom 6
5. $B \rightarrow \neg \neg A$ \hspace{1cm} 3, 4, HS
6. $\neg \neg A \rightarrow A$ \hspace{1cm} Axiom 5
7. $B \rightarrow A$ \hspace{1cm} 5, 6, HS
QED

4. b. 1. $A \rightarrow B$ \hspace{1cm} P
2. $B \rightarrow C$ \hspace{1cm} P
3. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ \hspace{1cm} Part (a)
4. $(A \rightarrow B) \rightarrow (A \rightarrow C)$ \hspace{1cm} 2, 3, MP
5. $A \rightarrow C$ \hspace{1cm} 1, 4, MP
QED.

d. 1. $\neg A \lor A$ \hspace{1cm} Part (c)
2. $(\neg A \lor A) \rightarrow (A \lor \neg A)$ \hspace{1cm} Axiom 3
3. $A \lor \neg A$ \hspace{1cm} 1, 2, MP
QED.
f.  1. $\neg A \rightarrow \neg \neg \neg A$  
    Part(e)
  2. $(\neg A \rightarrow \neg \neg \neg A) 
          \rightarrow ((A \lor \neg A) \rightarrow (A \lor \neg \neg \neg A))$  
    Axiom 4
  3. $(A \lor \neg A) \rightarrow (A \lor \neg \neg \neg A)$  
    1, 2, MP
  4. $A \lor \neg A$  
    Part (d)
  5. $A \lor \neg \neg \neg A$  
    3, 4, MP
  6. $(A \lor \neg \neg \neg A) \rightarrow (\neg \neg A \lor A)$  
    Axiom 3
  7. $\neg \neg A \rightarrow A$  
    5, 6, MP
  8. $\neg \neg A \rightarrow A$  
    7, Definition of $\rightarrow$
    QED.

h.  1. $B \rightarrow \neg \neg B$  
    Part (e)
  2. $(B \rightarrow \neg \neg B) 
          \rightarrow ((\neg A \lor B) \rightarrow (\neg A \lor \neg \neg B))$  
    Axiom 4
  3. $(\neg A \lor B) \rightarrow (\neg A \lor \neg \neg B)$  
    1, 2, MP
  4. $(\neg A \lor \neg \neg B) \rightarrow (\neg \neg B \lor \neg A)$  
    Axiom 3
  5. $(\neg A \lor B) \rightarrow (\neg \neg B \lor \neg A)$  
    3, 4, HS (i.e., Part (b))
  6. $(A \rightarrow B) \rightarrow (\neg B \lor \neg A)$  
    5, Definition of $\rightarrow$
    QED.

5. $C$ might have reasoned as follows: Since $A$ doesn’t know the color of his hat, it follows that the hats on $B$ and $C$ are not both red. After listening to $A$, $B$ says that he doesn’t know the color of his hat. So of course $A$ and $C$ don’t both have red hats. But $C$ can’t be wearing a red hat. For if $C$ were wearing a red hat, then $B$ would have been wearing a white hat and would have told the sheriff this fact. Therefore, $C$ must be wearing a white hat. He tells this to the sheriff and she sets him free.

Chapter 7

Section 7.1

1. b. $[p(0, 0) \lor p(1, 0)] \land [p(0, 1) \lor p(1, 1)]$.

2. b. $\exists x \ q(x)$, where $x \in \{0, 1\}$. d. $\exists y \ p(y, x)$, where $y \in \{0, 1\}$.

f. $\forall x \ p(x)$, where $x$ is a positive even natural number.

3. b. $x$, $y$, and $f(x)$ are all terms. Therefore, $p(y)$ and $q(f(x), y)$ are wffs. Thus $p(y) \rightarrow q(f(x), y)$ is a wff. It follows that $\exists x \forall y \ (p(y) \rightarrow q(f(x), y))$ is a wff.

5. b. The three occurrences of $y$, left to right, are bound, bound, and free. The single occurrence of $x$ is free.


9. b. W is false with respect to the interpretation.

10. b. The only eating that takes place is when a bird eats a worm.

11. b. Everything is a chocolate bar and some person eats everything.

12. b. \( \forall x \forall y (\neg (e(x, a) \lor e(y, a)) \rightarrow \exists z (d(z, x) \land d(z, y) )) \), where \( a = 0 \).

13. b. There are four possible interpretations, depending on the values assigned to \( p(a) \) and \( p(b) \). \( W \) is true when \( p(a) = p(b) \), and \( W \) is false when \( p(a) \neq p(b) \).

14. b. Any interpretation that makes \( p(x) \) true for every \( x \) in the domain.

15. b. Let the domain be \( \{a, b\} \), where \( p(a) = \text{True} \) and \( p(b) = \text{False} \). d. Let \( D = \{a, b\} \), \( p(a) = q(b) = \text{True} \) and \( p(b) = q(a) = \text{False} \). Then the antecedent is true and the consequent is false. f. Let \( D = \{d, e\} \). \( f(d) = d, f(e) = e \), let \( p \) be equality, and let \( y \) be \( d \). The wff is false with respect to this interpretation.

16. b. Let the domain be \( \{a, b\} \), where \( p(a) = \text{True} \) and \( p(b) = \text{False} \). Then \( p(a) \rightarrow p(b) \) is false. Thus \( W \) is invalid.

17. b. Let \( \{a, b\} \) be the domain. If \( p(a, a) = \text{False} \) or \( p(b, b) = \text{False} \), then \( W \) is true since the antecedent is false. Otherwise \( p(a, a) = p(b, b) = \text{True} \). In this case the consequent is true, which makes \( W \) true.

19. b. For any domain \( D \), suppose \( c \) is assigned the value \( d \in D \). If \( p(d) \) is true, then \( \exists x p(x) \) is also true. So the wff is true no matter what value is assigned to \( p(d) \). Therefore, any interpretation is a model. d. Let \( I \) be an interpretation for the wff, with domain \( D \). Assume \( I \) is a model for the antecedent. Then \( A(d) \land B(d) \) is true for some \( d \in D \). Therefore, \( A(d) \) is true for some \( d \in D \), and \( B(d) \) is true for some \( d \in D \). Thus \( I \) is a model for \( \exists x A(x) \) and \( I \) is a model for \( \exists x B(x) \). So \( I \) is a model for the consequent \( \exists x A(x) \land \exists x B(x) \). So the wff is valid. f. If the antecedent is true for a domain \( D \), then \( A(d) \rightarrow B(d) \) is true for all \( d \in D \). If \( A(d) \) is true for some \( d \in D \), then \( B(d) \) is also true by MP. Thus the consequent is true for \( D \).

20. b. Suppose the wff is satisfiable. Then there is an interpretation making \( \exists x (p(x) \land \neg p(x)) \) true. This says there is some \( d \) in the domain such that \( p(d) \land \neg p(d) \) is true, which is impossible.

21. Let \( x_1, \ldots, x_n \) be the free variables of \( W \) and let \( c_1, \ldots, c_n \) be the corresponding constants in \( c(W) \). Let \( I \) be an interpretation for \( W \) with domain \( D \), where each free variable \( x_i \) is assigned the value \( d_i \) in \( D \). We can define an interpretation \( J \) for \( c(W) \), where \( J \) is obtained from \( I \) by replacing each assignment of \( x_i \) to \( d_i \) with the assignment \( c_i \) to \( d_i \). Then \( W \) with respect to \( I \) is the same as \( c(W) \) with respect to \( J \). So if \( I \) is a model for \( W \), then \( J \) is a model for \( c(W) \). For the converse, just reverse the steps and start with an interpretation \( J \) for \( c(W) \)
23. To start things off we’ll show that if \( x \) is a free variable of \( W \), then \( W \) is unsatisfiable if and only if \( \exists x \ W \) is unsatisfiable. Suppose that \( W \) is unsatisfiable. Let \( I \) be an interpretation with domain \( D \) for the wff \( \exists x \ W \). If \( I \) is a model for \( \exists x \ W \), then \( W(x/d) \) is true with respect to \( I \) for some element \( d \in D \). This being the case, we can define an interpretation \( J \) for \( W \) by letting \( J \) be the same as \( I \) with the extra assignment of the free variable \( x \) to the element \( d \). Since \( W \) is unsatisfiable, it follows that \( W(x/d) \) false with respect to \( J \). But \( W(x/d) \) with respect to \( J \) is the same \( W(x/d) \) with respect to \( I \), which is true. This contradiction shows that \( I \) is not a model for \( \exists x \ W \). Therefore, \( \exists x \ W \) is unsatisfiable.

Suppose \( \exists x \ W \) is unsatisfiable. Let \( I \) be an interpretation with domain \( D \) for \( W \), where \( x \) is assigned the value \( d \in D \). Now define an interpretation \( J \) for \( \exists x \ W \) by letting \( J \) be obtained from \( I \) by removing the assignment of \( x \) to \( d \). Then \( J \) is an interpretation for the unsatisfiable wff \( \exists x \ W \). So \( W(x/e) \) is false with respect to \( J \) for all elements \( e \in D \). In particular, \( W(x/d) \) is false with respect to \( J \), and thus also with respect to \( I \). Therefore, \( I \) is not a model for \( W \). Therefore, \( W \) is unsatisfiable.

The preceding two paragraphs tell us that if \( x \) is free in \( W \), then \( W \) is unsatisfiable if and only if \( \exists x \ W \) is unsatisfiable. The proof now follows by induction on the number \( n \) of free variables in a wff \( W \). If \( n = 0 \), then \( W \) does not have any free variables, so \( W \) is it’s own existential closure. So assume that \( n > 0 \) and assume that part (1) is true any wff with \( k \) free variables, where for \( k < n \). Since \( \exists x \ W \) contains \( n − 1 \) free variables, it follows by induction that \( \exists x \ W \) is unsatisfiable if and only if its existential closure is unsatisfiable. But the existential closure of \( \exists x \ W \) is the same as the existential closure of \( W \). So it follows that \( W \) is unsatisfiable if and only the existential closure of \( W \) is unsatisfiable. QED.

Section 7.2

1. b. The left side is true for \( I \) iff \( A(d) \lor B(d) \) is true for some \( d \in D \) iff either \( A(d) \) or \( B(d) \) is true for some \( d \in D \) iff the right side is true for \( I \).

d. \( \forall x \forall y W(x, y) \) is true for \( I \) iff \( W(d, e) \) is true for all \( d, e \in D \) iff \( W(d, e) \) is true for all \( d, e \in D \) iff \( \forall y \forall x W(x, y) \) is true for \( I \).

2. b. The assumption that \( x \) is not free in \( C \) means that any substitution \( x/t \) does not change \( C \). In other words, \( C(x/t) = C \) for all possible terms \( t \). We’ll assume that \( J \) is an interpretation with domain \( D \). If \( I \) is a model for \( \exists x \ C \), then \( C(x/d) \) is true for \( I \) for some \( d \) in \( D \). Since \( C(x/d) = C \), it follows that \( C \) is true for \( I \). Therefore, \( I \) is a model for \( C \). If \( I \) is a model for \( C \), then \( C \) is true for \( I \). Since \( C = C(x/d) \) for all \( d \) in \( D \), it follows that \( C(x/d) \) is true for \( I \) for all \( d \) in \( D \) and thus also true for some \( d \) in \( D \). Therefore, \( I \) is a model for \( \exists x \ C \).

3. b. \( \exists x \ (C \to A(x)) \equiv \exists x \neg (C \lor A(x)) \equiv \neg C \lor \exists x \ A(x) \equiv C \to \exists x \ A(x) \).

d. \( \forall x \ (C \land A(x)) \equiv \neg \exists x \neg (C \land A(x)) \equiv \neg \exists x \neg (C \lor \neg A(x)) \equiv \neg (\neg C \lor \exists x \neg A(x)) \equiv C \land \neg \exists x \neg A(x) \equiv C \land \forall x A(x) \).

4. b. \( \forall x \forall y \exists z ((\neg p(x) \lor \neg q(y) \lor p(z)) \land (\neg p(x) \lor \neg q(y) \lor q(z))) \).
d. $\forall x (\neg p(x, f(x)) \lor p(x, y))$.

f. $\exists x \exists y \exists z \forall u \forall v ((p(x, y) \lor p(w, w) \lor \neg p(u, v) \lor \neg p(v, u))$
\hspace{2cm} \land (p(y, z) \lor p(w, w) \lor \neg p(u, v) \lor \neg p(v, u))$
\hspace{2cm} \land (\neg p(x, z) \lor p(w, w) \lor \neg p(u, v) \lor \neg p(v, u)))$.

5. b. $\forall x \forall y \exists z (\neg p(x) \lor \neg q(y) \lor (p(z) \land q(z)))$.

d. $\forall x (\neg p(x, f(x)) \lor p(x, y))$.

f. $\exists x \exists y \exists z \forall u \forall v ((p(x, y) \land p(y, z) \land \neg p(x, z))$
\hspace{2cm} \lor p(w, w) \lor \neg p(u, v) \lor \neg p(v, u))$.

6. b. $\exists x \forall y ((W(x) \rightarrow C) \land (C \rightarrow W(y)))$.

7. b. $\exists x (C(x) \land O(x))$. d. $\forall x (G(x) \rightarrow S(x))$.

8. b. $\exists x (B(x) \land \exists y (W(y) \land E(x, y))$.

d. $\exists x (B(x) \land \forall y (W(y) \rightarrow \neg E(x, y)))$.

f. $\forall x (B(x) \rightarrow \exists y (E(x, y) \land \neg W(y)))$.

9. b. $\exists x (P(x) \land \neg S(x)) \land \forall x (F(x) \rightarrow S(x)) \rightarrow \exists x (P(x) \land \neg F(x))$.

10. b. $\forall x \forall y (B(x) \land C(y) \rightarrow \neg K(x, y))$.

d. $\neg \forall x (P(x) \rightarrow G(x))$.

f. $\forall x (P(x) \land G(x) \rightarrow \forall y (A(y) \rightarrow K(x, y)))$.

Section 7.3

1. b. Line 3 is wrong because $y$ is free on line 2, which was constructed by EI. Therefore, line 2 can’t be used with the UG rule to generalize $y$.

d. Line 2 is wrong because $W(c) \neq W(x/c)$, where $W(c) = q(x, c)$. Thus EG can’t generalize to $x$ from $c$.

f. Line 2 is wrong because $y$ is not free to replace $x$. That is, the substitution of $y$ for $x$ yields a new bound occurrence of $y$. Thus UI can’t instantiate $y$ from $x$.

3. b. In step 4, EI must use a new constant different from $d$.

4. Notice that $x$ is free in the premise $p(x)$ on line 1 and this premise is used to infer $p(x)$ $\land q(x)$ on line 4. So UG cannot be used to generalize with respect to $x$. If we allowed such an inference, we would have proved the following invalid wff to be valid: $p(x) \land \forall x q(x)$ $\rightarrow$ $\forall x (p(x) \land q(x))$.

6. b. 1. $\forall x (p(x) \rightarrow q(x))$  \hspace{2cm} P

2. $\exists x p(x)$  \hspace{2cm} P

3. $p(c)$  \hspace{2cm} 2, EI

4. $p(c) \rightarrow q(c)$  \hspace{2cm} 1, UI

5. $q(c)$  \hspace{2cm} 3, 4, MP

6. $\exists x q(x)$  \hspace{2cm} 5, EG

QED  \hspace{2cm} 1–6, CP.
### d.
1. \( \forall x (p(x) \rightarrow q(x)) \)  
   \( P \)
2. \( \exists x p(x) \)  
   \( P \) [for \( \exists x p(x) \rightarrow \forall x q(x) \)]
3. \( p(c) \)  
   2, EI
4. \( p(c) \rightarrow q(c) \)  
   1, UI
5. \( q(c) \)  
   3, 4, MP
6. \( \exists x q(x) \)  
   5, EG
7. \( \exists x p(x) \rightarrow \exists x q(x) \)  
   2–6, CP
   QED  
   1, 7, CP.

### f.
1. \( \forall x (p(x) \rightarrow q(x)) \)  
   \( P \)
2. \( \forall x p(x) \)  
   \( P \) [for \( \forall x p(x) \rightarrow \forall x q(x) \)]
3. \( p(x) \)  
   2, UI
4. \( p(x) \rightarrow q(x) \)  
   1, UI
5. \( q(x) \)  
   3, 4, MP
6. \( \forall x q(x) \)  
   5, UG
7. \( \forall x p(x) \rightarrow \forall x q(x) \)  
   2–6, CP
   QED  
   1, 7, CP.

### h.
1. \( \exists x \forall y p(x, y) \)  
   \( P \)
2. \( \forall x (p(x, x) \rightarrow \exists y q(y, x)) \)  
   \( P \)
3. \( \forall y p(c, y) \)  
   1, EI
4. \( p(c, c) \)  
   3, UI
5. \( p(c, c) \rightarrow \exists y q(y, c) \)  
   2, UI
6. \( \exists y q(y, c) \)  
   4, 5, MP
7. \( q(b, c) \)  
   6, EI
8. \( \exists x q(x, c) \)  
   7, EG
9. \( \exists y \exists x q(x, y) \)  
   8, EG
   QED  
   1–9, CP.

### 7. b.
1. \( \forall x (p(x) \rightarrow q(x)) \)  
   \( P \)
2. \( \exists x p(x) \)  
   \( P \)
3. \( \neg \exists x q(x) \)  
   \( P \) [for \( \exists x q(x) \)]
4. \( p(c) \)  
   2, EI
5. \( p(c) \rightarrow q(c) \)  
   1, UI
6. \( q(c) \)  
   4, 5, MP
7. \( \exists x q(x) \)  
   6, EG
8. \( \text{False} \)  
   3, 7, Contr
9. \( \exists x q(x) \)  
   3–8, IP
   QED  
   1, 2, 9, CP.
\textbf{d.} 1. \(\exists x \forall y p(x, y)\) \(P\)
2. \(\forall x (p(x, x) \rightarrow \exists y q(y, x))\) \(P\)
3. \(\neg \exists y \exists x q(x, y)\) \(P\) [for \(\exists y \exists x q(x, y)\)]
4. \(\forall y p(c, y)\) 1, EI
5. \(p(c, c)\) 4, UI
6. \(p(c, c) \rightarrow \exists y q(y, c)\) 2, UI
7. \(\exists y q(y, c)\) 5, 6, MP
8. \(q(d, c)\) 7, EI
9. \(\exists x q(x, c)\) 8, EG
10. \(\exists y \exists x q(x, y)\) 9, EG
11. False 3–10, Contr
12. \(\exists y \exists x q(x, y)\) 3–11, IP
QED 1, 2, 12, CP.

\textbf{8. b.} Let \(C(x)\) mean \(x\) is a committee member, \(R(x)\) mean \(x\) is rich, \(F(x)\) mean \(x\) is famous, and \(O(x)\) mean \(x\) is old. Then the argument can be formalized as
\[\forall x (C(x) \rightarrow R(x) \land F(x)) \land \exists x (C(x) \land O(x)) \rightarrow \exists x (C(x) \land O(x) \land F(x)).\]
1. \(\forall x (C(x) \rightarrow R(x) \land F(x))\) \(P\)
2. \(\exists x (C(x) \land O(x))\) \(P\)
3. \(C(d) \land O(d)\) 2, EI
4. \(C(d) \rightarrow R(d) \land F(d)\) 1, UI
5. \(C(d)\) 3, Simp
6. \(R(d) \land F(d)\) 4, 5, MP
7. \(F(d)\) 6, Simp
8. \(C(d) \land O(d) \land F(d)\) 3, 7, Conj
9. \(\exists x (C(x) \land O(x) \land F(x))\) 8, EG
QED 1–9, CP.

\textbf{d.} Let \(Q(x)\) mean \(x\) is a rational number, and \(R(x)\) mean \(x\) is a real number. Then the argument can be formalized as
\[\forall x (Q(x) \rightarrow R(x)) \land \exists x Q(x) \rightarrow \exists x R(x).\]
1. \(\forall x (Q(x) \rightarrow R(x))\) \(P\)
2. \(\exists x Q(x)\) \(P\)
3. \(Q(c)\) 2, EI
4. \(Q(c) \rightarrow R(c)\) 1, UI
5. \(R(c)\) 3, 4, MP
6. \(\exists x R(x)\) 5, EG
QED 1–6, CP.
9. First prove the left side implies the right side, then the converse.

b. 1. \( \forall x (A(x) \land B(x)) \)  
2. \( A(x) \land B(x) \) 1, UI  
3. \( A(x) \) 2, Simp  
4. \( \forall x A(x) \) 3, UG  
5. \( B(x) \) 2, Simp  
6. \( \forall x B(x) \) 5, UG  
7. \( \forall x A(x) \land \forall x B(x) \) 4, 6, Conj  
QED 1–7, CP.

d. 1. \( \exists x (A(x) \rightarrow B(x)) \)  
2. \( A(d) \rightarrow B(d) \) 1, EI  
3. \( \forall x A(x) \)  
   \( P \) [for \( \forall x A(x) \rightarrow \exists x B(x) \)]  
4. \( A(d) \) 3, UI  
5. \( B(d) \) 2, 4, MP  
6. \( \exists x B(x) \) 5, UG  
7. \( \forall x A(x) \rightarrow \exists x B(x) \) 3–6, CP  
QED 1, 2, 7, CP.
11. 1. $\exists x \ (r(x) \land \forall y \ (p(y) \rightarrow q(x, y)))$  
2. $\forall x \ (r(x) \rightarrow \forall y \ (s(y) \rightarrow \neg q(x, y)))$  
3. $p(x)$  
4. $r(d) \land \forall y \ (p(y) \rightarrow q(d, y))$  
5. $\forall y \ (p(y) \rightarrow q(d, y))$  
6. $p(x) \rightarrow q(d, x)$  
7. $q(d, x)$  
8. $r(d) \rightarrow \forall y \ (s(y) \rightarrow \neg q(d, y))$  
9. $r(d)$  
10. $\forall y \ (s(y) \rightarrow \neg q(d, y))$  
11. $s(x) \rightarrow \neg q(d, x)$  
12. $\neg s(x)$  
13. $p(x) \rightarrow \neg s(x)$  
14. $\forall x \ (p(x) \rightarrow \neg s(x))$  

QED  

12. b. 1. $\exists x \ A(x)$  
2. $\exists x \ B(x)$  
3. $A(c)$  
4. $B(c)$  
5. $A(c) \land B(c)$  
6. $\exists x \ (A(x) \land B(x))$  

4. $r(d) \land \forall y \ (p(y) \rightarrow q(d, y))$  
5. $p(x) \rightarrow q(d, x)$  
6. $q(d, x)$  
7. $\neg s(x)$  
8. $p(x) \rightarrow \neg s(x)$  
9. $\forall x \ (p(x) \rightarrow \neg s(x))$  

QED  

13. b. A proof consists of the following six steps in both directions, along with the appropriate reasons.  

$d. \exists x \ (C \land A(x)); \ C \land A(c); \ C; \ A(c); \ A(x); \ C \land \exists x \ A(x).$

d. See proof of Exercise 9c. f. Similar to part (d). h. See proof of Exercise 9d.

14. b. Let $A \rightarrow B$ and $B \rightarrow C$ be valid wffs. Consider an arbitrary interpretation of these two wffs with domain $D$. Then $A \rightarrow B$ and $B \rightarrow C$ are true for $D$. Thus we can apply HS to conclude that $A \rightarrow C$ is true for $D$. Since the interpretation was arbitrary, it follows that $A \rightarrow C$ is valid.

15. b. The term $t$ is free to replace $x$ in $W(x)$.
16. Let \( n \) be the number of quantifiers and connectives in \( W(x) \). If \( n = 0 \), then \( W(x) \) is an atom. So \( W(t) \) is also an atom with all variables free. Therefore, \( A(t)I \) is obtained from \( A(t) \) by replacing all variables in \( t \) by constants in the domain \( D \) and all variables from \( A(x) \) (except \( x \)) by constants in \( D \). In other words, \( W(t)I = W(t)I \).

Now assume \( W(x) \) has \( n \) quantifiers and connectives and the statement to be proved is true for all wffs with less than \( n \) quantifiers and connectives. Consider the following cases. (1) If \( W(x) = \neg A(x) \), then \( W(t)I = (\neg A(t))I = \neg (A(t))I \). But the induction hypothesis tells us that \( \neg (A(t))I = \neg (A(t))I = W(t)I \). Therefore, \( W(t)I = W(t)I \). (2) if \( W(x) = A(x) \lor B(x) \), Then the induction hypothesis tells us that \( A(t)I = A(t)I \) and \( B(t)I = B(t)I \). So it follows that \( W(t)I = W(t)I \).

The other cases \( W(x) = A(x) \land B(x) \) and \( W(x) = A(x) \rightarrow B(x) \) are similar. (3) Assume \( W(x) = Qy A(x) \), where \( Q \) is either the existential or the universal quantifier. Since \( t \) is free to replace \( x \) in \( W(x) \), it follows that every variable in \( t \) is free in \( Qy A(t) \). So the induction hypothesis tells us that \( A(t)I = A(t)I \). Therefore, \( W(t)I = (Qy A(t))I = Qy (A(t)I) = (Qy A(t)I) = W(t)I \). The three cases cover all possibilities for the structure of \( W(x) \). QED.

### Chapter 8

#### Section 8.1

2. 1. \( t = u \)  
    2. \( u = v \)  
    3. \( (u = v) \land (t = u) \)  
    4. \( t = v \)

    \( P \)  
    \( P \)  
    1, 2, Conj  
    3, EE, where \( p(t, u) = "t = u" \) and \( p(t, v) = "t = v." \)

    QED  
    1–4, CP.

4. b. 1. \( t = u \)  
    2. \( p(... t ...) \)  
    3. \( q(... t ...) \)  
    4. \( p(... u ...) \)  
    5. \( q(... u ...) \)  
    6. \( p(... u ...) \land q(... u ...) \)

    \( P \)  
    \( P \)  
    \( P \)  
    1, 2, EE  
    1, 3, EE  
    4, 5, Conj

    QED  
    1–6, CP.

d. 1. \( x = y \)  
    2. \( \exists z p(... x ...) \)  
    3. \( p(... x ...)(z/c) \)  
    4. \( p(... y ...)(z/c) \)  
    5. \( \exists z p(... y ...) \)

    \( P \)  
    \( P \)  
    2, EI  
    1, 3, EE  
    4, EG

    QED  
    1–5, CP.
6. b. Proof of \( p(x) \rightarrow \forall y ((x = y) \rightarrow p(y)) \).

1. \( p(x) \)  
2. \( x = y \) \( P [\text{for } (x = y) \rightarrow p(y)] \)
3. \( p(y) \)  
4. \( (x = y) \rightarrow p(y) \)  
5. \( \forall y ((x = y) \rightarrow p(y)) \)  
\text{QED}  

Proof of \( \forall y ((x = y) \rightarrow p(y)) \rightarrow p(x) \):

1. \( \forall y (x = y \rightarrow p(y)) \)  
2. \( \neg p(x) \) \( P [\text{for } p(x)] \)
3. \( x = x \rightarrow p(x) \)  
4. \( x \neq x \)  
5. \( x = x \) \( \text{EA} \)
6. \( \text{False} \)  
7. \( p(x) \)  
\text{QED}  

7. b. \( \text{even}(x) = \exists z (x = 2z) \).

d. \( \exists z (a = bz + r) \land (0 \leq r < b) \).

8. b. One possible answer is
\[ \exists x \exists y (A(x) \land A(y) \land (x \neq y) \land \forall z (A(z) \rightarrow (z = x \lor z = y))) \]

9. b. Proof that (b) implies (a).

1. \( \exists x A(x) \land \forall x \forall y (A(x) \land A(y) \rightarrow (x = y)) \)  
2. \( \neg \exists x (A(x) \land \forall y (A(y) \rightarrow (x = y)) \) \( \text{P [for first wff]} \)
3. \( \forall x (\neg A(x) \lor \exists y (A(y) \land (x \neq y))) \)  
4. \( \exists x A(x) \)  
\text{1, Simp}  
5. \( A(c) \)  
6. \( \neg A(c) \lor \exists y (A(y) \land (c \neq y)) \)  
7. \( \exists y (A(y) \land (c \neq y)) \)  
\text{5, 6, DS}  
8. \( A(a) \land (c \neq a) \)  
\text{7, El}  
9. \( A(a) \)  
\text{8, Simp}  
10. \( \forall x \forall y (A(x) \land A(y) \rightarrow (x = y)) \)  
\text{1, Simp}  
11. \( A(c) \land A(a) \rightarrow (c = a) \)  
\text{10, UI, UI}  
12. \( A(c) \land A(a) \)  
\text{5, 9, Conj}  
13. \( c = a \)  
\text{11, 12, MP}  
14. \( c \neq a \)  
\text{8, Simp}  
15. \( \text{False} \)  
\text{13, 14, Contr}  
16. \( \exists x (A(x) \land \forall y (A(y) \rightarrow (x = y))) \)  
\text{2–15, IP}  
\text{QED}  
\text{1, 16, CP.}  

Section 8.2
2. b. 1. \{x < -b\} y := -b \{x < y\} \quad AA
2. \{-a < -b\} x := -a \{x < -b\} \quad AA
3. \{a > b\} x := -a \{x < -b\} \quad T
4. \{a > b\} x := -a \{x < -b\} \quad 2, 3, \text{ Consequence}
QED \quad 1, 4, \text{ Composition.}

3. Use the composition rule (8.12) applied to a sequence of three statements.

b. 1. \{y - x < x\} y := y - x \{y < x\} \quad AA
2. \{y - (y - x) < y - x\} x := y - x \{y - x < x\} \quad AA
3. 2x < y \Rightarrow y - (y - x) < y - x \quad T
4. \{2x < y\} x := y - x \{y - x < x\} \quad 2, 3, \text{ Conseq}
5. \{2x < y\} x := y - x ; y := y - x \{y < x\} \quad 1, 4, \text{ Comp}
6. \{2x < y + x\} y := y + x \{2x < y\} \quad AA
7. \quad x < y \Rightarrow 2x < y + x \quad T
8. \{x < y\} y := y + x \{2x < y\} \quad 6, 7, \text{ Conseq}
QED \quad 8, 5, \text{ Composition.}

4. b. First prove \{\text{True} \land x \neq y\} \ x := y \{x = y\}:

Proof: 1. \{y = y\} x := y \{x = y\} \quad AA
2. \quad \text{True} \land (x \neq y) \Rightarrow (y = y) \quad T
3. \quad \{\text{True} \land (x \neq y)\} x := y \{x = y\} \quad 1, 2, \text{ Consequence}
QED. 

Secondly prove \text{True} \land \neg (x \neq y) \Rightarrow x = y. \text{ This is a valid wff because} 
\neg (x \neq y) \equiv (x = y). \text{ Thus the original wff is correct, by the if-then rule.}

d. Prove the following two statements:
1. \{\text{True} \land (x > y)\} x := y + 1; y := x + 1 \{x \leq y\}
2. \quad \text{True} \land (x \leq y) \Rightarrow (x \leq y).

5. b. Use the if-then-else rule. Thus we must prove the two statements,
\{\text{True} \land (x < y)\} y := y - 1 \{x \leq y\} \quad \text{and} \quad \{\text{True} \land (x \geq y)\} x := -x; y := -y \{x \leq y\}.

For example, the second statement can be proved as follows:

Proof: 1. \{x \leq -y\} y := -y \{x \leq y\} \quad AA
2. \{-x \leq -y\} x := -x \{x \leq -y\} \quad AA
3. \text{True} \land (x \geq y) \quad P \text{ [for CP]}
4. \quad x \geq y \quad 3, \text{ Simp}
5. \quad -x \leq -y \quad 4, \ T
6. \text{True} \land (x \geq y) \Rightarrow (-x \leq -y) \quad 3-5, \text{ CP}
7. \quad \{\text{True} \land (x \geq y)\} x := -x \{x \leq -y\} \quad 2, 6, \text{ Consequence}
QED \quad 1, 7, \text{ Composition.}

6. b. The wff is incorrect when \(x = -1\) and \(y = -1\).
8. b. The postcondition $i = \text{floor}(x)$, is equivalent to $x < i + 1 \land i \leq x$. This statement has the form $Q \land \neg C$, where $C$ is the condition of the while loop and $Q$ is the suggested loop invariant. To show the while loop is correct with respect to $Q$, show $\{Q \land C\} i := i - 1 \{Q\}$ is correct. Once this is done, show that $\{x < 0\} i := -1 \{Q\}$ is correct.

10. Letting $Q$ denote the loop invariant, the while loop can be proved correct with respect to $Q$ by proving the following wff:

$$\{Q \land b \leq x\} x := x - b; y := y + 1 \{Q\}.$$

The parts of the program before and after the while loop can be proved correct by proving the following two wffs:

$$\{a \geq 0 \land b > 0\} x := a; y := 0 \{Q\}$$
$$\{Q \land \neg(b \leq x)\} r := x; q := y \{a = qb + r \land 0 \leq r < b\}.$$

12. The following program computes the ceiling of an arbitrary real number $x$, where the statements $S_1$ and $S_2$ are the two programs that follow:

$$\{\text{True}\} \text{ if } x \leq 0 \text{ then } S_1 \text{ else } S_2 \{i = \text{ceiling}(x)\}.$$

The program $S_1$ to compute the ceiling of a nonpositive real number $x$ uses the loop invariant $x \leq i$.

$$\{x \leq 0\}$$
$$i := 0;$$
$$\text{while } i \geq x + 1 \text{ do } i := i - 1$$
$$\{i = \text{ceiling}(x)\}.$$

The program $S_2$ to compute the ceiling of a positive real number $x$ uses the loop invariant $i - 1 \leq x$.

$$\{x > 0\}$$
$$i := 1;$$
$$\text{while } x > i \text{ do } i := i + 1$$
$$\{i = \text{ceiling}(x)\}.$$

The proof of correctness is similar to the answer to Exercise 8.

13. b. $\{16 = 16 \land (\text{if } j + 1 = i \text{ then } 16 \text{ else } d[j + 1] = 33)\}.$
14. b. 1. \{\text{odd}(\text{if } i - 1 = j \text{ then } a[i] + 1 \text{ else } a[i - 1])\}

\[ a[j] := a[i] + 1 \]

\{\text{odd}(a[i - 1])\}  

AAA

2. even(a[i]) \land (i = j + 1)  

\[ P \text{ [for CP]} \]

3. \(i = j + 1\)  

2, Simp

4. even(a[i])  

2, Simp

5. odd(a[i] + 1)  

4, \(T\)

6. odd(\text{if } i - 1 = j \text{ then } a[i] + 1 \text{ else } a[i - 1])  

3, 5, \(T\)

7. even(a[i]) \land (i = j + 1)  

\[ \rightarrow \text{odd(\text{if } i - 1 = j \text{ then } a[i] + 1 \text{ else } a[i - 1})} \]  

2–6, \(CP\)

QED  

1, 7, Consequence.

15. b. After applying AAA twice to the postcondition and two assignments we obtain the condition

\[ a[2] = (\text{if } 2 = a[2] \text{ then } 1 \text{ else } a[2]). \]

If we assume the precondition \(a[2] = 2\), then we obtain

\[ a[2] = (\text{if } 2 = a[2] \text{ then } 1 \text{ else } a[2]) = 1, \]

which gives \(a[2]\) two distinct values. Therefore, the precondition does not imply the obtained condition.

d. After applying AAA to the postcondition and assignment we obtain the condition

\[ \exists x \ (x = (\text{if } 2 = a[2] \text{ then } 1 \text{ else } a[2]) \land (x = a[2] \text{ then } 1 \text{ else } a[x]) = 1. \]

The precondition \(a[1] = 2 \land a[2] = 2\) implies that the preceding condition can be written as \(a[1] = 1\). Since we can’t have \(a[1]\) with two different values, it follows that the given precondition does not imply the obtained condition.

16. b. Define \(f(i, x) = x - i\). If \(s = (i, x)\), then after the execution of the loop body the state will be \(t = (i, x - 1)\). Thus \(f(s) = x - i\) and \(f(t) = x - 1 - i\). To prove termination, assume \(P\) and \(C\) are true and prove that \(f(s), f(t) \in \mathbb{N}\) and \(f(s) > f(t)\). So assume \(\text{int}(i) \land \text{int}(x) \land i \leq x\) and \(i < x\). It follows that \(i\) and \(x\) are integers and \(i < x\). So \(x - i\) is a positive integer and \(x - 1 - i\) is a non-negative integer. In other words, both \(x - i\) and \(x - 1 - i\) are natural numbers, which tells us that \(f(s), f(t) \in \mathbb{N}\). Since subtraction by 1 yields a smaller number we have \(x - i > x - 1 - i\), so that \(f(s) > f(t)\). Therefore, the loop terminates.

17. b. We are given \(f(x, y) = \max(x, y)\) and \(W = \mathbb{N}\). To prove termination, assume \(P\) and \(C\) are true and prove that \(f(s), f(t) \in \mathbb{N}\) and \(f(s) > f(t)\). So assume \(\text{pos}(x) \land \text{pos}(y)\) and \(x \neq y\). If \(s = (x, y)\), then the state \(t\) after the execution of the loop body has two possible values. If \(x < y\), then \(t = (x, y - x)\). In this case, since \(x > 0\) and \(y > 0\), it follows that \(f(s), f(t) \in \mathbb{N}\) and we have

\[ f(s) = f(x, y) = \max(x, y) = y > \max(x, y - x) = f(x, y - x) = f(t). \]
If $x > y$, then $t = (x - y, y)$. In this case, since $x > 0$ and $y > 0$, it follows again that $f(s)$, $f(t) \in \mathbb{N}$ and we have

$$f(s) = f(x, y) = \max(x, y) = x > \max(x - y, y) = f(x - y, y) = f(t).$$

Therefore, the loop terminates.

18. b. The definition $f(x, y) = \min(x, y)$ cannot be used because there are state values $s$ and $t$ such that $f(s) \leq f(t)$, which is contrary to the need in (8.20) for $f(s) > f(t)$. For example, if $s = (x, y) = (10, 13)$, then $f(s) = 10$. But after the body of the loop executes, we have $t = (x, y - x) = (10, 3)$, which gives $f(t) = 10$.

20. Let $P$ be the loop invariant $P = \text{“member}(a, x) = \text{member}(a, L)\text{”}$ and let $C$ be the while-loop condition $C = \text{“}x \neq (\ )\text{”}$ and $a \neq \text{head}(L)\text{”}$. The proof of partial correctness follows by composition from the correctness of the following three statements.

1. $\{\text{True}\} \ x := L \ P \}$.
2. $\{P\} \ \text{while } C \ \text{do } x := \text{tail}(x) \ \text{od} \ \{P \land \neg C\}$. This statement follows by the while-rule once we prove $\{P \land C\} \ x := \text{tail}(x) \ \{P\}$.
3. $\{P \land \neg C\} \ r := (x \neq (\ )) \ \{r = \text{member}(a, L)\}$.

The proofs are straightforward using properties of lists. One such property is that if $x \neq (\ )$ and $a \neq \text{head}(x)$ and $\text{member}(a, x) = \text{member}(a, L)$, then $\text{member}(a, X) = \text{member}(a, L)$. So $\text{member}(a, \text{tail}(x)) = \text{member}(a, L)$.

Proof of Termination.

Let $W = \mathbb{N}$ with the usual ordering and let $f(a, L, x) = \text{length}(x)$. If $s = (a, L, x)$, then the state $t$ after the execution of the loop body is $t = (a, L, \text{tail}(x))$. To prove termination, assume $P$ and $C$ are true and prove that $f(s), f(t) \in \mathbb{N}$ and $f(s) > f(t)$. So assume “$\text{member}(a, x) = \text{member}(a, L)$” and “$x \neq (\ )$ and $a \neq \text{head}(L)$”. It follows that $\text{length}(x) > 0$ and $\text{length}(\text{tail}(x)) \geq 0$, so we have $f(s), f(t) \in \mathbb{N}$. Since $\text{length}(x) > \text{length}(\text{tail}(x))$, it follows that $f(s) > f(t)$. Therefore, the loop terminates.

Section 8.3


2. b. $\exists S \ \exists A \ \exists B \ (\forall x (A(x) \lor B(x) \rightarrow S(x))) \land (\forall x (S(x) \rightarrow A(x) \lor B(x)))$.

3. b. Let $N$ stand for nation, $S$ for state, and $C$ for county. Then we can write the statement as $\exists N \ \exists S \ \exists C \ (N(S) \land S(C) \land (C = \text{Washington}))$. The wff is third order.

d. Let $C, N, S, Q, T, R$ stand for continent, nation, state, county, city, and street. Then we can write the statement as $\exists C \ \exists N \ \exists S \ \exists Q \ \exists T \ \exists R \ (C(N) \land N(S) \land S(Q) \land Q(T) \land T(R) \land Q = \text{Lincoln} \land R = \text{Broadway})$. The wff is sixth order.
4. The following wff assumes that $x$ varies over the natural numbers $\mathbb{N}$.

$$\forall S \left( \forall x \left( S(x) \rightarrow \mathbb{N}(x) \right) \land S(0) \land \forall x \left( S(x) \rightarrow S(\text{succ}(x)) \right) \right) \rightarrow \forall x \ S(x).$$

6. To see that the wff is satisfiable, we can choose an interpretation such that $p(x, y)$ is false for all $x, y \in D$. This makes the wff true.

To see that the wff is invalid, let $D = \{a, b\}$, let $p(x, y)$ be true for all $x, y \in D$. With this interpretation, the wff becomes

$$\exists x \ \exists y \left( \text{True} \rightarrow \forall Q \left( Q(x) \rightarrow \neg Q(y) \right) \right),$$

which is equivalent to

$$\exists x \ \exists y \ \forall Q \left( Q(x) \rightarrow \neg Q(y) \right).$$

So the wff has the following meaning. There are elements $x, y \in \{a, b\}$ such that for every subset $Q$ of $\{a, b\}$, $x \in Q$ implies $y \notin Q$. But this statement is false when $Q = \{a, b\}$. So the wff is false with respect to the interpretation. Therefore, the wff is invalid.

7. Think of $S(x)$ as $x \in S$. b. For any domain $D$ the consequent is true because $S$ can be $D$ itself. Thus the wff is true for all domains. d. For any domain the antecedent and consequent are both false for $S = \emptyset$. Thus the wff is true for all domains.

9. b. Let $l$ and $m$ be two lines intersecting at distinct points $x$ and $y$. It follows from Axiom 2 that $l = m$. QED. d. Assume the statement is false. Then there is a point, say $p$, which lies on every line. By Axiom 4 there are three distinct points $x, y$ and $z$, and any line on $x$ and $y$ is not on $z$. By Axiom 1 there are three lines, $L$ on $x$ and $y$, $M$ on $x$ and $z$, and $N$ on $y$ and $z$. By Axiom 4, $z$ is not on $L$. If $y$ were on $M$, then by Axiom 2, $L$ and $M$ would be equal, contradicting the fact that $z$ is not on $L$. So $y$ is not on $M$. Similarly, $x$ is not on $N$. Our assumption says that every line is on $p$. Therefore, $L, M$, and $N$ all lie on $p$. This says that $p$ is distinct from the three points $x, y$, and $z$. For otherwise, two of the three lines $L, M$, and $N$, would be equal. Since $L$ and $M$ are both on $p$ and $p \neq x$ it follows that $L = M$. But we know that $L \neq M$. This is a contradiction. Therefore, not every line lies on the same point. QED.
10. Here are some sample formalizations.

b. $\forall L \forall M (\exists x \exists y ((x \neq y) \land L(x) \land M(x) \land L(y) \land M(y)) \rightarrow (L = M))$.

Proof:
1. $\forall x \forall y ((x \neq y) \rightarrow \forall L \forall M (L(x) \land L(y) \land M(x) \land M(y) \rightarrow (L = M))$  
   Axiom 2
2. $\exists x \exists y ((x \neq y) \land L(x) \land M(x) \land L(y) \land M(y))$  
   $P$ [for CP]
3. $(a \neq b) \land L(a) \land M(a) \land L(b) \land M(b)$  
   2, EI, EI
4. $a \neq b$  
   3, Simp
5. $(a \neq b) \rightarrow \forall L \forall M (L(a) \land L(b) \land M(a) \land M(b) \rightarrow (L = M))$  
   1, UI, UI
6. $\forall L \forall M (L(a) \land L(b) \land M(a) \land M(b) \rightarrow (L = M))$  
   4, 5, MP
7. $L(a) \land L(b) \land M(a) \land M(b) \rightarrow (L = M)$  
   6, UI, UI
8. $L(a) \land L(b) \land M(a) \land M(b)$  
   3, Simp
9. $L = M$  
   7, 8, MP
10. $\exists x \exists y ((x \neq y) \land L(x) \land M(x) \land L(y) \land M(y)) \rightarrow (L = M)$  
    2–9, CP
11. $\forall L \forall M (\exists x \exists y ((x \neq y) \land L(x) \land M(x) \land L(y) \land M(y)) \rightarrow (L = M))$  
    10, UG, UG
QED.

d. $\forall x \exists L \neg L(x)$.

Proof:
1. $\neg \forall x \exists L \neg L(x)$  
   $P$ [for $\forall x \exists L \neg L(x)$]
2. $\exists x \forall L L(x)$  
   1, $T$
3. $(a \neq b) \land (a \neq c) \land (b \neq c)$  
   $\land \forall L (L(a) \land L(b) \rightarrow \neg L(c))$  
   Axiom 4, EI, EI, EI
4. $l(a) \land l(b)$  
   3, Simp, Axiom 1, EI
5. $m(a) \land m(c)$  
   3, Simp, Axiom 1, EI
6. $n(b) \land n(c)$  
   3, Simp, Axiom 1, EI
7. $\neg l(c)$  
   3, Simp, UI, 4, MP
8. $\neg m(b)$  
   3, 4, 5, 7, equality
9. $\neg n(a)$  
   3, 4, 6, 7, equality
10. $\forall L L(d)$  
    2, EI
11. $l(d) \land m(d) \land n(d)$  
    8, UI (3 times), Conj
12. $(d \neq a) \land (d \neq b) \land (d \neq c)$  
    7, 8, 9, 11, equality
13. $m = l$  
    4, 5, 11, 12, Axiom 2
14. $l(c)$  
    5, 13, equality
15. False  
    7, 14, Contr
QED
1–15, IP.
Chapter 9

Section 9.1

1. b. \((A \lor C \lor \neg E \lor F) \land (A \lor D \lor \neg E \lor F) \land (B \lor C \lor \neg E \lor F)\).

d. \(\forall y \neg p(f(y), y) \lor q(b)\).

f. \(\forall x [(\neg p(x, f(x)) \lor r(x, f(x), g(x))) \land (q(x, g(x)) \lor r(x, f(x), g(x)))\].

3. b. 1. \(p \lor q\)  2. \(\neg p \lor r\)  3. \(\neg r \lor \neg p\)  4. \(\neg q\)  5. \(q \lor r\)  6. \(r\)  7. \(\neg p\)  8. \(q\)  9. \(\Box\)  

QED.

4. b. \{y/x\}.  d. \{x/f(b), y/a, z/b\}.  e. \{x/f(a), y/f(b), z/b\}.

5. b. No mgu.  d. No mgu.

6. b. No mgu.

8. b. 1. \(p(u, v)\)  2. \(q(w, z)\)  3. \(\neg p(y, f(x, y)) \lor \neg p(f(x, y), f(x, y)) \lor \neg q(x, f(x, y))\)  4. \(\neg p(f(x, y), f(x, y)) \lor \neg q(x, f(x, y))\)  5. \(\neg q(x, f(x, y))\)  6. \(\Box\)  

QED.

d. Resolve the two clauses by unifying four atoms: \(p(x)\) and \(p(f(a))\) from the clause \(p(x) \lor p(f(a))\), and \(p(y)\) and \(p(f(z))\) from the clause \(\neg p(y) \lor \neg p(f(y))\). The four atoms can be unified by mgu \{x/f(a), y/f(a), z/a\}, which yields the common value \(p(f(a))\). Apply the mgu to both clauses to obtain the two clauses \(p(f(a)) \lor p(f(a))\) and \(\neg p(f(a)) \lor \neg p(f(a))\). Now delete all occurrences of \(p(f(a))\) from the first clause and delete all occurrences of \(\neg p(f(a))\) from the second clause. This leaves the empty clause.
9. b. After negating the statement and putting the result in clausal form we obtain the following proof:

1. \( \neg p \lor q \)  
2. \( \neg q \lor r \)  
3. \( p \)  
4. \( \neg r \)  
5. \( q \)  
6. \( \neg q \)  
7. \( \square \)

QED.

d. After negating the statement and putting the result in clausal form we obtain the following proof:

1. \( \neg A \lor \neg B \lor C \)  
2. \( \neg A \lor B \)  
3. \( A \)  
4. \( \neg C \)  
5. \( B \)  
6. \( \neg A \lor \neg B \)  
7. \( \neg A \)  
8. \( \square \)

QED.

10. b. After negating the statement and putting the result in clausal form we obtain the following proof:

1. \( \neg p(x) \lor q(x) \)  
2. \( p(a) \)  
3. \( \neg q(z) \)  
4. \( q(a) \)  
5. \( \square \)

QED.

d. After negating the statement and putting the result in clausal form we obtain the following proof:

1. \( p(a, y) \)  
2. \( \neg p(x, x) \lor q(f(x), x) \)  
3. \( \neg q(w, z) \)  
4. \( q(f(a), a) \)  
5. \( \square \)

QED.

11. b. This follows because \( x \epsilon = x \) for any variable \( x \). d. Using part (1) of (9.5) repeatedly
we have for any variable $x$ the equations $x((\theta \sigma)\alpha) = (x(\theta \sigma))\alpha = ((x(\theta))\sigma)\alpha = (x\theta)(\sigma\alpha) = x(\theta(\sigma\alpha))$. Therefore, $(\theta \sigma)\alpha = \theta(\sigma\alpha)$.

12. b. In first order predicate calculus the argument can be written as the wff $A \rightarrow B$ is valid, where $A$ and $B$ are defined as follows:

$$\begin{align*}
A &= \forall x (B(x) \rightarrow \neg L(x)) \land \forall x (C(x) \rightarrow \neg D(x)) \land \forall x (\neg L(x) \rightarrow D(x)), \\
B &= \forall x (B(x) \rightarrow \neg C(x)),
\end{align*}$$

where $B(x)$ means “$x$ is a baby,” $L(x)$ means “$x$ is logical,” $D(x)$ means “$x$ is despised,” and $C(x)$ means “$x$ can manage a crocodile.” After negating the wff and transforming the result into clausal form we obtain the proof:

1. $\neg B(x) \lor \neg L(x)$  
2. $\neg C(y) \lor \neg D(y)$  
3. $L(z) \lor D(z)$  
4. $B(a)$  
5. $C(a)$  
6. $\neg L(a)$  
7. $D(a)$  
8. $\neg C(a)$  
9. $\square$  

QED.

13. b. Let $C(x)$ mean “$x$ is a committee member,” $R(x)$ mean “$x$ is rich,” $F(x)$ mean “$x$ is famous,” and $O(x)$ mean “$x$ is old.” Then the argument can be formalized as

$$\forall x (C(x) \rightarrow R(x) \land F(x)) \land \exists x (C(x) \land O(x)) \rightarrow \exists x (C(x) \land O(x) \land F(x)).$$

After negating the wff and transforming the result into clausal form we obtain the proof:

1. $\neg C(x) \lor R(x)$  
2. $\neg C(y) \lor F(y)$  
3. $C(a)$  
4. $O(a)$  
5. $\neg C(z) \lor \neg O(z) \lor \neg F(z)$  
6. $F(a)$  
7. $\neg O(a) \lor \neg F(a)$  
8. $\neg F(a)$  
9. $\square$  

QED.
d. Let \( Q(x) \) mean “\( x \) is a rational number” and \( R(x) \) mean “\( x \) is a real number.” Then the argument can be formalized as
\[
\forall x (Q(x) \rightarrow R(x)) \land \exists x Q(x) \rightarrow \exists x R(x).
\]
After negating the wff and transforming the result into clausal form we obtain the proof:

1. \( \neg Q(x) \lor R(x) \)  

2. \( Q(a) \)  

3. \( \neg R(y) \)  

4. \( R(a) \)  

5. \( \square \)  

QED.

14. b. The clausal form of \( W \) is \( \forall x (p(x, f(x)) \lor \neg p(f(x), f(x))) \). Now define an interpretation with domain \( D = \{0, 1\} \), where \( f(0) = 1, f(1) = 0 \), and \( p(x, y) \) means \( x = y \). The wff becomes \( ((0 = 1) \lor \neg (1 = 1)) \land ((1 = 0) \lor \neg (0 = 0)) \), which is false. So the clausal form is invalid.

15. b. Applying Skolem’s rule to \( W \) we obtain the wff \( (p(a) \rightarrow C) \land (p(b) \land \neg C) \). Define an interpretation for this wff by letting \( C = \text{False} \), \( p(a) = \text{False} \), and \( p(b) = \text{True} \). This interpretation makes the wff true. So it is satisfiable.

d. After applying Skolem’s algorithm correctly to \( W \) (i.e., remove \( \rightarrow \) and move \( \neg \) inward before applying Skolem’s rule), we obtain the following clausal form: \( \forall x ((\neg p(x) \lor C) \land (p(b) \land \neg C)) \). Let \( I \) be an interpretation for this wff. If \( C \) is true for \( I \), then \( (p(b) \land \neg C) \) is false, so the wff is false. If \( C \) is false for \( I \), then the wff becomes \( \forall x ((\neg p(x) \lor \text{False}) \land (p(b) \land \neg \text{False})) \equiv \forall x (\neg p(x) \land p(b)), \) which is false. Therefore, the wff is false for \( I \). Since \( I \) was arbitrary, the wff is unsatisfiable.

Section 9.2

1. b. \( \text{isGrandChildOf}(x, y) \leftrightarrow \text{isParentOf}(y, z), \text{isParentOf}(z, x). \)

2. b. The following definition will work if \( x \neq y \):
\[
\text{isCousinOf}(x, y) \leftrightarrow \text{isChildOf}(x, z), \text{isChildOf}(y, w), \text{isSibling}(z, w).
\]

4. b. \( \{g^n(a) | n \in \mathbb{N}\} \cup \{g^n(b) | n \in \mathbb{N}\}. \)

5. b. The program takes an infinite walk.

6. b. If \( r \) is defined over the set \( \{a, b, c, d\} \), then the reflexive closure \( rc \) can be defined
by the following five clauses.

\[
\begin{align*}
\text{rc}(x, y) &\leftarrow r(x, y). \\
\text{rc}(a, a). \\
\text{rc}(b, b). \\
\text{rc}(c, c). \\
\text{rc}(d, d).
\end{align*}
\]

7. b. \(\text{pcat}(\langle \rangle, y, y).\)
\(\text{pcat}(a :: t, y, a :: z) \leftarrow \text{pcat}(t, y, z).\)

8. b. \(\text{member}(x, x :: t).\)
\(\text{member}(x, y :: t) \leftarrow \text{member}(x, t).\)

d. \(\text{makeSet}(\langle \rangle, \langle \rangle).\)
\(\text{makeSet}(x :: t, x :: u) \leftarrow \text{all}(x, t, y), \text{makeSet}(y, u).\)

f. \(\text{equalSets}(x, y) \leftarrow \text{subset}(x, y), \text{subset}(y, x).\)

h. \(\text{equalBags}(x, y) \leftarrow \text{subBag}(x, y), \text{subBag}(y, x).\)

Chapter 10

Section 10.1

2. b. False; True; True is its own inverse.

4. b. The operation is commutative if the transpose of the table equals the original table. Another way to say this is that the triangles created by the Northwest to Southeast diagonal are reflections of each other.

5. b. d.

\[
\begin{array}{cccc}
\circ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a & d & c \\
c & c & d & a & b \\
d & d & c & b & a \\
\end{array}
\]
6. b. Notice that \((a \circ b) \circ a \neq a \circ (b \circ a)\).

d. 

8. b. \(f(f(f(f(x)))) = f(f(g(x))) = g(g(x)) = x\).

9. b. Show that \(y^{-1} \circ x^{-1}\) is an inverse of \(x \circ y\). I.e., show \((x \circ y) \circ (y^{-1} \circ x^{-1}) = e\).

Since \((x \circ y)^{-1}\) is also the inverse of \(x \circ y\), the result follows from (10.2).

10. The tables for \(+_5\) and \(\cdot_5\) are given in Figure 10.3. Associativity and commutativity follow from the properties of regular addition and multiplication of natural numbers. The tables show that 0 is the identity for \(+_5\), and 1 is the identity for \(\cdot_5\), and each element has an inverse with respect to \(+_5\), and each nonzero element has an inverse with respect to \(\cdot_5\). For example, \(2 \cdot_5 3 = 3 \cdot_5 2 = 1\), which shows that 2 and 3 are inverses with respect to \(\cdot_5\).

Section 10.2

2. b. \(x(x + y) = xx + xy = x + xy = x\), by part (a).

Alternatively, \(x(x + y) = (x + 0)(x + y) = x + 0y = x + 0 = x\).

d. \(x(x + y) = x\overline{x} + xy = 0 + xy = xy\).

4. b. Use (10.7) twice to obtain \(\overline{x} + \overline{y} + xy\overline{z} = \overline{x} + \overline{y} + x\overline{z} = \overline{x} + \overline{y} + \overline{z}\).

Expand the right side to obtain the same expression:
\(\overline{x} + \overline{y} + xy\overline{z} = \overline{x} + \overline{y} + \overline{x} + \overline{y} + \overline{z} = \overline{x} + \overline{y} + \overline{z}\).

5. b. 0. d. \(x + y\). f. \(xy\). h. \(x + y\). j. \(xy\). l. \(xy\).

6. b.

\[
\begin{array}{ccc}
\circ & a & b \\
\hline
a & a & b \\
b & b & b \\
\end{array}
\quad
\begin{array}{ccc}
\circ & a & b \\
\hline
a & b & a \\
b & b & b \\
\end{array}
\]

d.

7. b. \(x + y\). d. \((x + y)\). f. \((z + \overline{y} + \overline{x})(x + z + \overline{x})\).

8. \(1 + x = (\overline{x} + x) + x = \overline{x} + (x + x) = \overline{x} + x = 1\).
10. Show that \( B \) consists of distinct pairs of elements of the form \( x \) and \( \bar{x} \). Let \( x \in B \). If \( \bar{x} = x \), then we have \( x = x + x = x + \bar{x} = 1 \) and \( x = xx = x\bar{x} = 0 \). This contradicts the fact that \( 0 \neq 1 \). Therefore, \( B \) consists of pairs of distinct elements.

11. b. Since \( x \leq y \) means \( x = xy \), it will suffice to show that \( x = xy \) iff \( y = x + y \). If \( x = xy \), then we have \( x + y = xy + y = xy + 1y = (x + 1)y = 1y = y \). So if \( x = xy \), then \( y = x + y \). Now assume that \( y = x + y \). Then we have \( xy = x(y + x) = xx + xy = x + xy = x(1 + y) = x1 = x \). So if \( y = x + y \), then \( x = xy \).

12. We’ll show that \( \text{glb}(x, y) = xy \). We must show that \( xy \) is a lower bound of \( \{x, y\} \), and that \( xy \) is the greatest of all lowerbounds of \( \{x, y\} \). Since we can write \( xy = (xx)y = x(xy) = x(yx) = (xy)x \), it follows that \( xy \leq x \). Similarly we have \( xy = xyy \). So \( xy \leq y \). Thus \( xy \) is a lower bound of \( \{x, y\} \). Now if \( c \) is any lower bound of \( \{x, y\} \), then \( c = cx \) and \( c = cy \). Thus \( c = cx = (cy)x = c(yx) = c(xy) \). So \( c \leq xy \). Therefore, \( xy = \text{glb}(x, y) \). The proof that \( \text{lub}(x, y) = x + y \) is similar.

### Section 10.3

2. Let \( \exp(0, 0) = 0 \), \( \exp(s(x), 0) = 1 \), and \( \exp(x, s(y)) = \text{mult}(x, \exp(x, y)) \) where \( s(x) \) denotes the successor of \( x \).

4. Let the strings of \( A^* \) be represented by lists of elements in \( \text{Lists}(A) \). Correspondingly, the empty string \( \Lambda \) is represented by \( \langle \rangle \). Then we set append = cons, isemptyS = isEmptyL, headS = head, and tailS = tail. The axioms of the string algebra are clearly satisfied.

6. \( \text{flatten}(l) = \)
   
   if isEmpty(l) then \( l \)
   
   else if isAtom(head(l)) then head(l) :: flatten(tail(l))
   
   else concatenate(flatten(head(l)), flatten(tail(l))).

8. \( \text{bin}(4) = \text{addQ}(0, \text{bin}(2)) = \text{addQ}(0, \text{addQ}(0, \text{bin}(1))) = \text{addQ}(0, \text{addQ}(0, \text{addQ}(1, \text{emptyQ}))) = \text{addQ}(0, \text{addQ}(0, \langle 1 \rangle)) = \text{addQ}(0, \langle 1, 0, 0 \rangle) = \langle 1, 0, 0 \rangle. \)

10. In equational form we have \( \text{postorder}(<\text{emptyTree}> = \text{emptyQ} \) and \( \text{postorder}(<\text{tree}(L, x, R)> = \text{addQ}(x, \text{apQ}(\text{postorder}(L), \text{postorder}(R))). \)

12. Reverse (the elements in a queue).

14. \( \text{isEmptyQ}(<\text{emptyQ}> = \text{isEmptyL}(<\text{emptyL}>) = \text{True}. \)

\[
\text{isEmptyQ}(\text{addQ}(a, q)) = \begin{cases}
\text{isEmptyL}(\text{putLast}(a, q)) & \text{if isEmptyL}(q) \text{ then consL}(a, q) \\
\text{consL}(\text{headL}(q), \text{putLast}(a, \text{tailL}(q))) & \text{else}
\end{cases}
\]

\[
= \begin{cases}
\text{if isEmptyL}(q) \text{ then isEmpty(consL}(a, q) &) \\
\text{else isEmptyL}(\text{consL}(a, q), \text{putLast}(a, \text{tailL}(q))) &)
\end{cases}
\]

\[
= \text{False}.
\]
delQ(addQ(a, q)) = tailL(putLast(a, q))
= tailL(if isEmptyL(q) then consL(a, q)
else consL(headL(q), putLast(a, tailL(q))))
= if isEmptyL(q) then tailL(consL(headL(q), putLast(a, tailL(q))))
else tailL(consL(headL(q), putLast(a, tailL(q))))
= if isEmptyL(q) then q else putLast(a, tailL(q))
= if isEmptyQ(q) then q else addQ(a, deleQ(q)).

16. We need the equality relation on A.

17. b. Let P(x) denote the statement “plus(x, y) = add(x, y) for all y ∈ N.” Certainly P(0) is true because plus(0, y) = y = add(0, y) from the two definitions. So assume that P(x) is true, and prove that P(s(x)) is true. Starting with the expression plus(s(x), y) in P(s(x)), we obtain the other expression as follows:

\[
\begin{align*}
\text{plus}(s(x), y) &= s(\text{plus}(p(s(x)), y)) & \text{(definition of plus)} \\
&= s(\text{plus}(x, y)) & \text{(since } p(s(x)) = x) \\
&= \text{plus}(x, s(y)) & \text{(from part (a))} \\
&= \text{add}(x, s(y)) & \text{(induction)} \\
&= \text{add}(p(s(x)), s(y)) & \text{(since } p(s(x)) = x) \\
&= \text{add}(s(x), y) & \text{(definition of add)}. \\
\end{align*}
\]

So P(s(x)) is true. QED.

19. We’ll use induction on the length of z. The base case for z = emptyQ becomes apQ(x, apQ(y, emptyQ)) = apQ(x, y) = apQ(apQ(x, y), emptyQ). For the induction case, assume the equation is true for all queues z having length n, and show the equation true for the queue addQ(a, z) having length n + 1. As in the solution to Exercise 18, we’ll use the notation z:a for addQ(a, z). Starting with the left side of the equation we have:

\[
\begin{align*}
apQ(x, apQ(y, z:a)) &= apQ(x, apQ(y:front(z:a), delQ(z:a))) & \text{(def of apQ)} \\
&= apQ(apQ(x, y:front(z:a)), delQ(z:a)) & \text{(induction)} \\
&= apQ(apQ(x, y:front(z:a), delQ(z:a))) & \text{(Exercise 18)} \\
&= apQ(apQ(x, y), z:a) & \text{(def of apQ)}. \\
\end{align*}
\]

Section 10.4

1. b. \{(a, M), (a, N), (b, M)\}.

d. \{(a, #, M), (a, *, N), (b, #, M), (a, %, N), (b, *, N), (b, %, M)\}.

3. b. project(select(Program, Type, Movie), \{Station\}).

d. project(select(Rooms, Place, CH301), \{Seats\}).

f. Let A = join(Rooms, Schedule) and let
\[B = select(select(select(A, Dept, CS), Course, 252), Section, 2))\].

Then the desired expression is proj(B, Computer).
4. b. The attribute set for $R \triangleright \triangleleft R$ is equal to the attribute set for $R$. Call this attribute set $I$. So $t \in R \triangleright \triangleleft R$ iff there exist $r, s \in R$ such that $t(a) = r(a) = s(a)$ for all $a \in I$ iff $t \in R$.

6. b. Let $f = \text{pairs}$, where

$$\text{pairs} = \text{null} @ 1 \rightarrow \sim(\ ); \text{apndl} @ [[\text{hd} @ 1, \text{hd} @ 2], \text{pairs} @ [\text{tl} @ 1, \text{tl} @ 2]].$$

7. b. For any object $x$ we have $1 @ \sim(a, b) : x = 1 : (a, b) = a = \sim : x$. Similarly, $2 @ \sim(a, b) : x = 2 : (a, b) = b = \sim : x$.

9. It’s easy to see that slow and fast are equal when given the base inputs 0 and 1. Now assume slow @ sub1 = fast @ sub1 and slow @ sub2 = fast @ sub2, and assume the input is larger than 1. Starting with fast we obtain the following equations:

\begin{align*}
\text{fast} &= 1 @ g \\
&= 1 @ (\text{eq}0 \rightarrow \sim(0, 1); [2, +] @ g @ \text{sub1}) \\
&= 2 @ g @ \text{sub1} \quad \text{(def of g when input > 1)} \\
&= 2 @ (\text{eq}0 \rightarrow \sim(0, 1); [2, +] @ g @ \text{sub1}) @ \text{sub1} \quad \text{(def of g)} \\
&= 2 @ [2, +] @ g @ \text{sub1} @ \text{sub1} \quad \text{(def of g when input > 0)} \\
&= + @ g @ \text{sub2} \quad \text{(FP algebra)}.
\end{align*}

Now we’ll expand slow as follows:

\begin{align*}
\text{slow} &= + @ \text{slow} @ \text{sub1}, \text{slow} @ \text{sub2} \quad \text{(def of slow when input > 1)} \\
&= + @ \text{fast} @ \text{sub1}, \text{fast} @ \text{sub2} \quad \text{(induction)} \\
&= + @ [1 @ g @ \text{sub1}, 1 @ g @ \text{sub2}] \quad \text{(def of fast)} \\
&= + @ [1 @ [2, +] @ g @ \text{sub1} @ \text{sub1}, 1 @ g @ \text{sub2}] \quad \text{(def of g)} \\
&= + @ [2 @ g @ \text{sub2}, 1 @ g @ \text{sub2}] \quad \text{(FP algebra)} \\
&= + @ [2, 1] @ g @ \text{sub2} \quad \text{(FP algebra)} \\
&= + @ g @ \text{sub2} \quad (+ @ [2, 1] = +).
\end{align*}

Therefore, the two sides are equal.

Section 10.5

2. b. $x = 43$. d. 11. f. 81.

4. The proof left off with the statement that $n$ divides $a^i(a^j - 1)$. The hypothesis $\gcd(a, n) = 1$ implies that $\gcd(a^i, n) = 1$. So it follows from (2.2d) that $n$ divides $(a^j - 1)$, which means $a^j \equiv 1 \pmod{n}$. QED.

5. b. $e = 37$ works.

7. b. No. $1 +_3 1 = 2 \notin \{1, 4, 5\}$.

8. b. $\{0, 3, 6, 9\}$. 

10. \( f(\Lambda) = \text{length}(\Lambda) = 0 \) and if \( x \) and \( y \) are arbitrary strings in \( A^* \), then
\[
f(\text{cat}(x, y)) = \text{length}(\text{cat}(x, y)) = \text{length}(x) + \text{length}(y) = f(x) + f(y).
\]

12. Define the function \( f : \{a, b, c\} \rightarrow \mathbb{N} \) by \( f(a) = 1 \), \( f(b) = 0 \), and \( f(c) = 2 \). The table for \( \circ \) is symmetric about the main diagonal, which says \( \circ \) is commutative. Now we need to show that \( f(x \circ y) = f(x) + f(y) \) for all \( x, y \in \{a, b, c\} \). There are nine equations to check. For example,
\[
f(a \circ c) = f(b) = 0 = 1 + 2 = f(a) + f(c).
\]

13. b. \( \{ab^n \mid n \in \mathbb{N}\} \). d. \( \{a\} \).

### Chapter 11

#### Section 11.1

1. b. \( \{a, bc\} \). d. \( \{c, a, ab, abb, \ldots, ab^n, \ldots\} \). f. \( \{ac\} \cup \{a^n b c^m \mid m, n \in \mathbb{N}\} \).

2. b. \( a(a + b + c) \). d. \( a(aa)^* \). f. \( a^* + b^* + c^* \). h. \( (aa)^* + b(bb)^* \).

4. b. \( (aaa + aab + aba + ab + bab + bba + bbb)^* \)
   or \( ((a + b)(a + b)(a + b))^* \)
   d. \( b^*ab*(b^*ab^*a^*)^* \).

5. b. \( aa(b^* + a) \).

6. b. We can write \( a^* = \Lambda + aa^* \). Using this substitution we can write the left side as follows:
\[
a^*(b + ab^*) = (\Lambda + aa^*)(b + ab^*)
= b + ab^* + aa*b + aa^*ab^*
= b + (\Lambda + aa^*)ab^* + aa*b (\text{combine 2nd and 4th terms})
= b + a^*ab^* + aa^*b
= b + aa^*(b^* + b)
= b + aa^*b^*.
\]

8. b. \( \Lambda \). d. \( R^* \), where \( R \) is the regular expression formed by the sum of the letters in \( A \).

9. b. \( (b + ab + aab)^*(\Lambda + a + aa) \).

10. Let \( R = a^* \), \( S = \Lambda + a \), and \( T = \Lambda + aa \).

11. b. \( a^*ab \) and \( a^*ab + b \).
Section 11.2

2. b. States 0 (start), 1 (final), 2, and 3. \( T(0, a) = 1, T(0, b) = 2, T(2, c) = 1 \), and all other transitions go to state 3. d. States 0 (start), 1 (final), 2 (final), and 3. \( T(0, a) = 1, T(1, b) = 1, T(0, c) = 2 \), and all other transitions go to state 3.

f. States 0 (start), 1, 2 (final), 3 (final), 4, and 5. \( T(0, a) = 1, T(1, c) = 2, T(0, b) = T(1, b) = T(3, c) = T(4, b) = 3, T(1, a) = T(4, a) = 4 \), and all other transitions go to state 5.

4. \( T(0, a) = \{0, 2\}, T(0, \Lambda) = \{1\}, T(1, b) = \{1, 2\}, T(2, a) = \{3\}, \) and \( T(3, \Lambda) = \{2\} \), where all other transitions go to \( \varnothing \). State 0 is the start state and 3 is the final state.

5. b. States 0 (start) and 1 (final). \( T(0, a) = \{0, 1\}, T(0, b) = \{0\} \), and all other transitions go to \( \varnothing \).

6. b.

![Diagram](image1)

d.

![Diagram](image2)

8. b. \( a(b + aa^*b)^*aa^*b \).

10. We’ll label the states 0 (start), 5, 10, 15, 20, 25, 30, 35, and 40 to denote the sum of coins necessary to reach each state. Any additional coins inserted at states 20, 25, 30, 35, and 40 will be automatically returned. The Mealy machine can be described by the following graph:
Section 11.3

1. b. The NFA has eight states, 0 (start), 1, 2, 3, 4, 5, 6, 7 (final). $T(0, \Lambda) = \{1, 7\}$, $T(1, \Lambda) = \{2, 4\}$, $T(2, a) = \{3\}$, $T(3, \Lambda) = \{6\}$, $T(4, b) = \{5\}$, $T(5, \Lambda) = \{6\}$, $T(6, \Lambda) = \{1, 7\}$, and all other transitions map to $\emptyset$.

3. b. $T(0, a) = \{1\}$, $T(0, \Lambda) = \{1, 2\}$, $T(1, b) = T(2, a) = \{2\}$, where all other transitions going to $\emptyset$. State 0 is the start state and 2 is the final state.

d. Use the answer to part (c) to draw the picture.

4. b. States $\{0\}$ (start), $\{1, 2\}$ (final) and $\emptyset$, where $T_D(\{0\}, a) = T_D(\{1, 2\}, b) = \{1, 2\}$, and all other transitions map to state $\emptyset$.

5. b. The NFA has eight states, 0 (start), 1, 2, 3, 4, 5, 6, 7 (final). $T(0, \Lambda) = \{1, 7\}$, $T(1, \Lambda) = \{2, 4\}$, $T(2, a) = \{3\}$, $T(3, \Lambda) = \{6\}$, $T(4, b) = \{5\}$, $T(5, \Lambda) = \{6\}$, $T(6, \Lambda) = \{1, 7\}$, and all other transitions map to $\emptyset$. The DFA has three states, 0 (start, final), 1 (final), 2 (final). $T_D(0, a) = T_D(1, a) = T_D(2, a) = 1$, $T_D(0, b) = T_D(1, b) = T_D(2, b) = 2$.

6. The states are $\{0, 1, 4\}$ (start), $\{3\}$, and $\{2, 5\}$ (final).


9. b. The equivalence pair is $\{2, 3\}$. Therefore, the states are $\{0\}$, $\{1\}$, $\{2, 3\}$, $\{4\}$, where $\{0\}$ is the start state with final states $\{1\}$ and $\{2, 3\}$. $T_{\min}(\{0\}, a) = [1]$, $T_{\min}(\{0\}, b) = T_{\min}(\{2\}, a) = [2]$, $T_{\min}(\{1\}, a) = T_{\min}(\{1\}, b) = T_{\min}(\{4\}, a) = T_{\min}(\{4\}, b) = [4]$. 
10. b. The NFA has 9 states. It transforms into a DFA with three states, 0 (start), 1 (final), and 2. $T_0(0, a) = T_0(1, a) = T_0(2, a) = 1$, $T_0(0, b) = T_0(1, b) = T_0(2, b) = 2$. The minimum state DFA has two states, 0 (start) and 1 (final). $T_{\text{min}}(0, a) = T_{\text{min}}(1, a) = 1$ and $T_{\text{min}}(0, b) = T_{\text{min}}(1, b) = 0$. d. See exercise 5b for the eight state NFA and the three state DFA. The minimum state DFA has the single state 0 (start, final), and $T_{\text{min}}(0, a) = T_{\text{min}}(0, b) = 0$.

12. A DFA with all final states accepts the language $A^*$, where $A$ is the alphabet of the DFA. For example, if $A = \{a, b, c\}$, then the DFA accepts the language of the regular expression $(a + b + c)^*$. Just minimize the DFA and notice that the result is a single state DFA.

Section 11.4

1. b. $S \rightarrow a | bc$. d. $S \rightarrow aB | c$, $B \rightarrow \Lambda | bB$. f. $S \rightarrow ac | bC | aA$, $C \rightarrow \Lambda | cC$, $A \rightarrow bC | aA$.

h. $S \rightarrow aab | baaS | bbS$.

2. b. $S \rightarrow aa | ab | ac$.

d. $S \rightarrow a | aaS$.

f. $S \rightarrow \Lambda | aA | bB | cC$, $A \rightarrow \Lambda | aA$, $B \rightarrow \Lambda | bB$, $C \rightarrow \Lambda | cC$.

h. $S \rightarrow A | B$, $A \rightarrow \Lambda aaaA$, $B \rightarrow b | bbB$.

3. b. $S \rightarrow \Lambda aT | bT$, $T \rightarrow aU | bU$, $U \rightarrow aS | bS$.

d. $S \rightarrow bX | aY$, $X \rightarrow bX | aY$, $Y \rightarrow bY | Z$, $Z \rightarrow \Lambda | bL | aM$, $L \rightarrow bL | aM$, $M \rightarrow bK | aN$, $K \rightarrow bK | aN$, $N \rightarrow \Lambda | bN | Z$.

4. b. $S \rightarrow Ta$, $T \rightarrow \Lambda | Tab$.

6. $S$ is the start state with $I$ and $J$ final states. The transition function $T$ has values: $T(S, a) = T(I, b) = I$, $T(S, b) = T(J, a) = J$.

7. b. Let $L = \{w \mid w \in \{a, b\}^* \text{ and } w \text{ is a palindrome of even length}\}$ and suppose that $L$ is regular. Using the pumping lemma (11.13) we’ll choose $s = a^nbba^m$. Then $s$ can be written as $s = a^nbba^m = xyz$, where $y \neq \Lambda$ and $|xy| \leq m$. It follows that $y = a^i$ for some $i > 0$. Then $xz = a^{m-i}ba^m$, which is not in $L$. This contradicts the pumping lemma result that $xy^2z \in L$ for all $k \geq 0$. Thus $L$ is not regular.

d. Let $L = \{a^nb^k \mid n, k \in \mathbb{N} \text{ and } n \geq k\}$ and suppose that $L$ is regular. Using the pumping lemma (11.13) we’ll choose $s = a^nbba^m$. Then $s$ can be written as $s = a^nbba^m = xyz$, where $y \neq \Lambda$ and $|xy| \leq m$. It follows that $y = a^i$ for some $i > 0$. Then $xz = a^{m-i}ba^m$, which is not in $L$. This contradicts the pumping lemma result that $xy^2z \in L$ for all $k \geq 0$. Thus $L$ is not regular.
f. Let \( L = \{ a^p \mid p \text{ is a prime number} \} \) and suppose that \( L \) is regular. Using the pumping lemma (11.13) let \( s = a^p \) where \( p \) is a prime and \( p > m \). Then there are strings \( x, y, \) and \( z \), where \( y \neq \Lambda \) and such that \( s = xyz \), \( |xy| \leq m \), and \( xy^kz \in L \) for all \( k \geq 0 \). For \( k = 0 \) we have \( xz \in L \), which says \( xz = a^q \) for some prime \( q \). Since \( y \neq \Lambda \), it follows that \( y = a^i \) for some \( i > 0 \). Letting \( k = q \) we have \( xy^qz \in L \). But it follows that \( |xyqz| = qi + q = q(i + 1) \), which is not a prime number. This forces us to conclude that \( xy^qz \not\in L \). This contradiction tells us that \( L \) is not regular. Note: Another way to proceed is to pump \( y \) up \( p \) times and find a contradiction.

10. b. This is clear from part (a).

d. If \( \{ a^nba^n \mid n \in \mathbb{N} \} \) is regular, then by (11.16) \( f^{-1}(\{ a^nba^n \mid n \in \mathbb{N} \}) \) is regular. So by (11.15) and part (b) we must conclude that \( \{ a^nbc^n \mid n \in \mathbb{N} \} \) is regular. So by (11.16) and part (c) it follows that \( \{ a^n b^n \mid n \in \mathbb{N} \} \) is regular. But this is not the case. So \( \{ a^n ba^n \mid n \in \mathbb{N} \} \) can’t be regular.

Chapter 12

Section 12.1

1. b. \( S \to Tbb, T \to aTb \mid \Lambda \). d. \( S \to aSa \mid bSb \mid a \mid b \). f. \( S \to aSbS \mid bSaS \mid \Lambda \).

2. b. \( S \to AB, A \to aAb \mid \Lambda, B \to aBbb \mid \Lambda \). d. \( S \to \Lambda \mid aS \mid B, B \to ab \mid aBb \) (or alternatively, \( S \to aSb \mid aS \mid \Lambda \).)

Section 12.2

1. b. A PDA for the set of strings over \( \{ a, b \} \) with the same number of \( a \)'s and \( b \)'s has start state 0 and final state 1 with \( \bot \) as the starting stack symbol.

\[
\begin{align*}
(0, a, a, \text{push}(a), 0) \\
(0, a, b, \text{pop}, 0) \\
(0, a, \bot, \text{push}(a), 0) \\
(0, b, a, \text{pop}, 0) \\
(0, b, b, \text{push}(b), 0) \\
(0, b, \bot, \text{push}(b), 0) \\
(0, \Lambda, \bot, \text{nop}, 1)
\end{align*}
\]
d. A PDA to recognize the language of odd length palindromes has start state 0 and final state 2 with ⊥ as the starting stack symbol:

\begin{align*}
(0, a, ?, \text{push}(a), 0) \\
(0, b, ?, \text{push}(b), 0) \\
(0, a, ?, \text{nop}, 1) \\
(0, b, ?, \text{nop}, 1) \\
(1, a, a, \text{pop}, 1) \\
(1, b, b, \text{pop}, 1) \\
(1, \Lambda, \bot, \text{nop}, 2).
\end{align*}

3. b. State 0 is the start state and states 0 and 4 are final states with ⊥ as the starting stack symbol.

\begin{align*}
(0, a, \bot, \text{push}(a), 1) \\
(1, \Lambda, a, \text{push}(a), 2) \\
(2, a, a, \text{push}(a), 1) \quad \text{Push second } a. \\
(2, b, a, \text{pop}, 3) \\
(3, b, a, \text{pop}, 3) \\
(3, \Lambda, \bot, \text{nop}, 4).
\end{align*}

4. b. Let the two states be 0 and 1, where 0 is both the start state and the final state, and ⊥ is the starting stack symbol. The PDA can be represented as follows:

\begin{align*}
(0, a, \bot, \text{nop}, 1) \\
(1, a, \bot, \langle \text{push}(Y), \text{push}(a) \rangle, 1) \\
(1, a, a, \langle \text{pop, push}(Y), \text{push}(a) \rangle, 1) \\
(1, b, a, \langle \text{pop, pop} \rangle, 1) \\
(1, b, Y, \text{pop}, 1) \\
(1, b, \bot, \text{push}(X), 0).
\end{align*}

7. b. (0, a, a, \text{pop}, 0) \\
(0, b, b, \text{pop}, 0) \\
(0, \Lambda, S, \text{pop}, 0) \\
(0, \Lambda, S, \langle \text{pop, push}(b), \text{push}(S), \text{push}(a) \rangle, 0) \\
(0, \Lambda, S, \langle \text{pop, push}(S), \text{push}(a), \text{push}(a) \rangle, 0).

10. Any PDA to accept the language must make a decision whenever the first b is encountered after a string of \(n a\)'s. The decision must be to either accept \(n b\)'s or \(2n b\)'s without any further knowledge. Thus the decision is nondeterministic.
Section 12.3

1. b. \( S \to aS \mid b \). d. \( S \to aSc \mid B, B \to bBc \mid \Lambda \).

3. b. \( S \to abC, C \to A \mid cA, A \to aA \mid \Lambda \). The grammar is LL(1).

4. b. \( S \to abT, T \to aaST \mid \Lambda \). The grammar is LL(1).

6. b. The sentential forms that can occur in a rightmost derivation take the following forms, where \( n \geq 0 \): \( b^nSb^n, a^nS^n \). In each case, the handle is completely determined without scanning past it. So the grammar is LR(0).

7. The sentential forms that can occur in a rightmost derivation take the following two forms, where \( n \geq 0 \): \( a^nS b^n, a^n b^n \). The handle \( aSb \) can be determined with no lookahead beyond it. But the handle \( \Lambda \) is determined by observing the first \( b \) after an \( a \), or by observing the \( $ \). Thus the grammar is LR(1).

Section 12.4

1. b. \( S \to aAB \mid aB \mid aA \mid a, A \to aAb \mid ab, B \rightarrow Bb \mid b. \)

2. b. \( S \to TR \mid BR \mid c, T \rightarrow AC \mid AB, R \rightarrow BR \mid c, A \rightarrow a, B \rightarrow b, C \rightarrow TB. \)

3. b. \( S \rightarrow bAB \mid bA, A \rightarrow bBACT \mid bACT \mid bBAC \mid aT \mid a, T \rightarrow aT \mid a, B \rightarrow bB \mid b, C \rightarrow a. \)

4. b. Let \( z = a^m b^{m+1} c^{m+2} = uvwxy \). Consider two cases: (1) There is at least one \( a \) in either \( v \) or \( x \). (2) Neither \( v \) nor \( x \) contain any \( a \)'s. In case (1) no \( c \) can occur in \( vwx \) because \( |vwx| \leq m \). So the pumped string \( uv^3 w^3 x^3 y \) will contain at least \( m + 3 \) \( a \)'s but only \( m + 2 \) \( c \)'s. Thus \( uv^3 w^3 x^3 y \) can’t be in the language. In case (2) \( v \) or \( x \) must contain \( b \)'s or \( c \)'s. Thus the string \( uv^3 w^3 x^3 y = uwy \) contains \( m \) or fewer \( b \)'s or it contains \( m \) or fewer \( c \)'s. Thus \( uwy \) can’t be in the language. These contradictions show that the language is not context-free.

5. b. This is clear from Part (a).
Chapter 13

Section 13.1

2. If the current cell does not contain #, then write an X, and scan left and right, writing an X each time, until # is found. Erase the X’s and go back to the # cell. Let 0 be the start state.

(0, #, #, S, Halt)      Bingo!
(0, Λ, X, L, 1)

(1, #, #, R, 3)     Found it going left.
(1, Λ, X, R, 2)     Now look right.
(1, X, X, L, 1)     Skip.

(2, #, #, L, 4)     Found it going right.
(2, Λ, X, L, 1)     Now look left.
(2, X, X, R, 2)

(3, X, Λ, R, 3)     Clean up to the right.
(3, Λ, Λ, L, 3)     Now go back left.
(3, #, #, S, Halt)      Done.

(4, X, Λ, L, 4)     Clean up to the left.
(4, Λ, Λ, R, 4)     Now go back right.
(4, #, #, S, Halt)      Done.

4. b. The leftmost string is moved right one cell position, overwriting the # symbol. Then the machine scans left and halts with the tape head at the leftmost cell of the number. Here is an informal algorithm:

    if current cell is Λ then
        move right and overwrite the # with Λ;
        move right and halt
    else
        write Λ
        move right to # and write 1
        move to left end of number and halt
    fi.
Here is the Turing machine code:

- \((0, \Lambda, \Lambda, R, 1)\)  
  - Left number is zero.
- \((0, 1, \Lambda, R, 2)\)  
  - Write \(\Lambda\) and move right.
- \((1, \#, \Lambda, R, \text{Halt})\)  
  - Write \(\Lambda\) and halt.
- \((2, 1, 1, R, 2)\)  
  - Scan right.
- \((2, \#, 1, L, 3)\)  
  - Write 1 and go left.
- \((3, \Lambda, \Lambda, R, \text{Halt})\)  
  - Found left end.
- \((3, 1, 1, L, 3)\)  
  - Scan left.

5. **b.** Assume tape head is at right end of input string. The machine will overwrite the input string with the answer and halt with the tape head at the left end of the answer. The start state is 0.

- \((0, 0, 0, L, 1)\)  
  - Skip the rightmost bit to add two.
- \((0, 1, 1, L, 1)\)
- \((1, 0, 1, L, 3)\)  
  - Done with add, move to left end.
- \((1, 1, 0, L, 2)\)  
  - Go to carry state.
- \((1, \Lambda, 1, S, \text{Halt})\)  
  - Done with add.
- \((2, 0, 1, L, 3)\)  
  - Done with add, move to left end.
- \((2, 1, 0, L, 2)\)  
  - Stay in carry state.
- \((2, \Lambda, 1, S, \text{Halt})\)  
  - Done with add.
- \((3, 0, 0, L, 3)\)  
  - Move left.
- \((3, 1, 1, L, 3)\)  
  - Move left.
- \((3, \Lambda, \Lambda, R, \text{Halt})\)  
  - Done with add.
6. First check for the empty string. Then perform the following loop: Overwrite the leftmost letter of the left string with X and go do the same for the corresponding letter of the right string.

(0, a, X, R, 1)  Go find a.
(0, b, X, R, 5)  Go find b.
(0, #, #, R, 7)  Done with left string.
(0, λ, λ, R, 8)  String is empty.

(1, a, a, R, 1)  Skip.
(1, b, b, R, 1)  Skip.
(1, #, #, R, 2)  

(2, a, X, L, 3)  Found a.
(2, b, 0, S, Halt)  Not equal.
(2, X, X, R, 2)  Skip.
(2, λ, 0, S, Halt)  Not equal.

(3, X, X, L, 3)  Skip.
(3, #, #, L, 4)  Skip.

(4, a, a, L, 4)  Skip.
(4, b, b, L, 4)  Skip.
(4, X, X, R, 0)  Go start again.

(5, a, a, R, 5)  Skip.
(5, b, b, R, 5)  Skip.
(5, #, #, R, 6)  

(6, a, 0, S, Halt)  Not equal.
(6, b, X, L, 3)  Found b.
(6, X, X, R, 6)  Skip.
(6, λ, 0, S, Halt)  Not equal.

(7, a, 0, S, Halt)  Not equal.
(7, b, 0, S, Halt)  Not equal.
(7, X, X, R, 7)  Skip.
(7, λ, 1, S, Halt)  They’re equal!

(8, a, 0, S, Halt)  Not equal.
(8, b, 0, S, Halt)  Not equal.
(8, #, #, R, 8)  
(8, λ, 1, S, Halt)  Equal empty strings!
8. Assume the tape head is pointing at the right end of the string. The machine will halt with the tape head at the left end of the successor string.

(0, Λ, a, S, halt) For input Λ, write a and halt.
(0, a, b, L, 1) Change rightmost a to b and go halt.
(0, b, c, L, 1) Change rightmost b to c and go halt.
(0, c, a, L, 0) Change rightmost c’s to a’s.

(1, Λ, Λ, R, halt) Move to left end of output string.
(1, a, a, L, 1)
(1, b, b, L, 1)
(1, c, c, L, 1).

Section 13.2

1. b. Z := 0; C := Y; while C ≠ 0 do Z := Z + X; C := pred(C) od.
   d. A := X; B := 1;
      while A ≠ 0 do S; A := 0; B := 0 od;
      while B ≠ 0 do T; B := 0 od.
   f. Use the fact that “X < Y” is equivalent to “Y monus X ≠ 0.”
   h. Notice that absoluteDiff(X, Y) = (X monus Y) + (Y monus X).

j. Z := X * Y; while Z ≠ 0 do S; Z := X * Y od. A solution that does not use multiplication can be written:

   A := X; B := Y;
   while A ≠ 0 do
     if B ≠ 0 then
       S;
       A := X; B := Y
     else
       A := 0
     fi
   od.

2. b. GE(x, y) can be defined by any of the the following expressions.

   GE(x, y) = monus(1, greater(x, y)).
   GE(x, y) = greater(succ(x), y)).
   GE(x, y) = sign(monus(succ(x), y)).

3. b. f is not defined at zero. Otherwise f is the predecessor function. d. If x = 0 then f(x, y) = 0. If x is divisible by y, then f(x, y) = x/y. Otherwise f is undefined.
4. Assume that $X$ is the input variable and $Y$ is the output variable. Then the following program computes $f(X)$.

$$Y := 0; \ Z := g(X, \ Y); \ \textbf{while} \ Z \neq 0 \ \textbf{do} \ Y := \text{succ}(Y); \ Z := g(X, \ Y) \ \textbf{od}.\]  

5. b. $a^{|\cdot|}$

6. b. $b \rightarrow \Lambda$ (halt). d. $ab \rightarrow ba, \ ac \rightarrow ca, \ bc \rightarrow cb, \ b \rightarrow bb$ (halt).

7. b. $X \rightarrow 1X$ (halt). d. $XbY \rightarrow YbbX$ (halt). f. $bX \rightarrow X$ (halt), $aX \rightarrow a*X, \ cX \rightarrow c*X, \ X*aY \rightarrow Xa*Y, \ X*cY \rightarrow Xc*Y, \ X*bY \rightarrow XY$ (halt), $X* \rightarrow X$ (halt).

8. b. $X \rightarrow aXa$ and $X \rightarrow bXb$ and $X \rightarrow cXc$ with axioms $a$, $b$, and $c$. d. Axioms $\{0, 1\}$; Inference rules $1X \rightarrow 1X1, \ 1X \rightarrow 1X0$. 
Chapter 14

Section 14.1

3. b. \( g \) is not computable because \( f_n(n) \) may not be defined. d. \( g \) is computable, but no contradiction arises if \( g \) is assumed to be in the list. So we can’t conclude \( g \) is not in the list.

4. Start a clock to count to \( k \) units of time, and simultaneously start the evaluation of \( f_n \) on input \( m \). If the evaluation stops before the clock reaches \( k \), then halt and return 1. Otherwise halt when the clock reaches \( k \) and return 0.

6. b. The sequence 1,3 will produce the equality \( aba = aba \). d. The sequence 2, 1, 1, 3 will produce the equality \( 01111110 = 01111110 \).

Section 14.2

2. b. \[
S \rightarrow aba \mid aTba \\
T \rightarrow abA \mid aTbA \\
Ab \rightarrow bA \\
Aa \rightarrow aa.
\]

3. b. \[
S \rightarrow aba \mid aTBa \\
T \rightarrow abA \mid aTBA \\
AB \rightarrow AX \rightarrow BX \rightarrow BA \\
bB \rightarrow bb \\
Aa \rightarrow aa.
\]

Section 14.3

2. Let \( l_1 \land l_2 \land \ldots \land l_n \) be a conjunction of \( n \) literals. The wff will be satisfiable if it doesn’t contain an occurrence of a variable \( x \) and its negation \( \neg x \). We can test for this property by first comparing \( l_i \) with the literals in the set \( \{l_2, \ldots, l_n\} \). If \( l_i \) is a variable \( x \), then we are looking for \( \neg x \). If \( l_i \) has the form \( \neg x \), then we are looking for \( x \). If no pair \( \{x, \neg x\} \) is found, then compare \( l_2 \) with the literals in the set \( \{l_3, \ldots, l_n\} \), and so on. In the worst case, this process takes \( n(n-1)/2 \) comparisons. Therefore, the 1-satisfiability problem is in \( P \).

4. b. \( A \) is in \( P \). d. \( A \) is \( NP \)-complete.

5. a. \( P \neq NP \). b. \( P \neq NP \) and all \( NP \)-complete problems are intractable. c. \( P = NP \).